Multi-Period Dual Pricing Algorithm in Non-Convex ISO Markets

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Abstract— Generation costs in electricity markets are non-convex functions of output; as a result, prices are not monotonically non-decreasing with demand. Consequently, it is difficult to define a single price at each node that results in a balance of supply and demand, while covering all generator operating costs. Therefore, most markets currently pay a two-part price at each node, consisting of a public marginal price and a private make-whole payment tailored to each generator who would otherwise incur variable costs that exceed their revenue. The expense of these make-whole payments, also called uplift payments, is usually allocated evenly across all customers. This allocation method does not take into consideration who benefits from the additional costs. This paper proposes an alternative algorithm for prices in a non-convex market and a means to allocate those prices to market participants called the Dual Pricing Algorithm. Basic principles of market design are used as the foundation for the new approach in an auction market that is revenue neutral and non-confiscatory. The general framework presents a cost allocation scheme that maintains the market surplus and can be further modified to consider equity objectives defined by the system operator.

***Index Terms*—Centralized day-ahead electricity market, power system economics, non-convex pricing, mixed integer linear programming.**

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Views expressed are not necessarily those of the FERC.

I. Nomenclature

A. Sets

D	Demand (i)
G	Generators (i)
T	Time horizon (t), T_r is total run time of a generator, T_1 is a generator's startup period

B. Generator Parameters

p_{it}^{max}	Maximum operating level for generator i in t
p_{it}^{min}	Minimum operating level for generator i in t
c_{it}	Marginal operating cost for generator i in t
c_{it}^{SU}	Fixed costs to startup generator i in t
c_{it}^{OC}	Fixed costs to operate generator i in t

C. Generator Variables

p_{it}	Cleared energy for generator i in t
u_{it}	Operating status of unit generator i in t
z_{it}	Startup variable for generator i in t
Π_i	Linear profit for generator i

D. Demand Parameters

b_{it}	Marginal value of demand i in t
d_{it}^{max}	Maximum demand for demand i in t

E. Demand Variables

d_{it}	Cleared demand for segment for demand i in t
Ψ_i	Net value received by demand i
λ_t	Public marginal price in t

F. DPA Variables

λ_t^{DPA}	Dual pricing algorithm LMP
$\lambda_t^{up/dn}$	Conditioning for an up/down deviation for λ_t^{DPA}
u_{it}^{p}	Uplift payment (credit) to generator i in t
u_{it}^{c}	Uplift charge (debit) to generator i in t
u_{it}^{pd}	Uplift payment (credit) to demand i in t
u_{it}^{cd}	Uplift charge (debit) to demand i in t

G. Additional Parameters

$\square^*$	Optimal solution for (1)
$c^{up/dn}$	Cost of an up/down deviation from λ_t^*

II. INTRODUCTION

In many commodity markets, supply and demand curves provide a single market clearing price. In electricity markets, the non-convexities in bid and offer functions can make the traditional single market clearing price insufficient for generators to recover costs (e.g., fixed startup costs). Markets in the U.S. currently provide make-whole or uplift payments for generators to ensure that they will at minimum recover their operating costs. Unlike the single market clearing price in convex markets, U.S. electricity markets pay a two-part price: a single price in time and space and a discriminatory non-public uplift payment. The allocation of the uplift payment to customers varies by market and is often evenly distributed among market participants, even though not all participants contribute to the need for such a payment [1]. When costs are allocated too broadly they dilute the price and location signal needed to stimulate investment in better alternatives. The outcome of the spot market has implications for both bilateral and investment markets. For multiple markets to be efficient, they must signal each other via public or transparent information.

Day-ahead markets aim to find the surplus maximizing generator commitment schedule, a mixed integer linear program called unit commitment (UC). After the efficient dispatch has been determined, a pricing run determines the price of electricity at each hour and node (or bus) in the market. Pricing practice by most independent system operators (ISO) reruns the unit commitment model fixing the binary on/off decisions and relaxing the minimum operating level of the fast-start generators to zero. The locational marginal prices (LMPs) result from the dual variable or shadow price of the node balance constraint. The uplift payments are determined ex-post based on the total losses of a generator. The LMPs are public and non-discriminatory information, while the uplifts are discriminatory and private, lest they divulge specific generator information. This public-private split means market participants and investors only know part of the information necessary to enter the market, resulting in a weak investment signal.

In addition to transparency issues, allocation of the uplift has caused poor investment signals. Historical examples in the upper peninsula of Michigan and on Cape Cod show that traditional pricing mechanisms can hide or misallocate funds [2], [3]. A pricing mechanism should follow basic economic principles in order to create an efficient market. With the underlying principle of maximizing social welfare and building on the description in [4], the proposed dual pricing algorithm outlined in this paper aims to provide an alternate approach to efficient prices and cost allocation of make-whole payments. The algorithm is based on the dual formulation to the post-unit commitment problem, hence it is called the Dual Pricing Algorithm. Unlike other pricing mechanisms, this algorithm allocates all costs, maintains market surplus, is non-confiscatory and revenue neutral. These principles are examined in detail in Section III. Section IV discusses previous literature. Section V explains the formulation the multi-period dual pricing algorithm. The results, discussion, and conclusions are in VI, VII, and VIII.

III. Fundamental Economic Principles

The basic principle of market design underlying the Dual Pricing Algorithm (DPA) is efficiency, as measured by the maximization of market surplus. From this basic principle, three other guiding principles are developed below. Day-ahead unit commitment determines the efficient schedule and dispatch for resources in electric markets. Because of the non-convexities, the market clearing price is not guaranteed to cover the startup and fixed operating costs for any individual generator. In order to guarantee that both generation and demand are not incentivized to leave the market (have non-negative profit and value), we include non-confiscation as the first of three principles for the DPA. Non-confiscation ensures that both suppliers and demand will at least break even if they are part of the efficient dispatch. The second principle is revenue neutrality, which implies revenue adequacy in the market. Specifically, we propose that the market should give out what it takes in. Third, the market should incentivize efficient participation and investment; this principle must hold in order to adequately build resources that will improve overall market efficiency. This should apply to both generation and transmission investments.

Volatility and a constant (increasing) price-demand relationship are two issues that arise in non-convex pricing literature [5], [6]. To economists, the volatility of efficient prices is of little concern. Prices should reflect the relationship between supply and demand, and should include any volatility due to congestion, scarcity, or non-convexities. However, electric markets often suppress volatility in favor of 'stable' prices or fixing a scarcity problem with an out-of-market correction. A pricing mechanism with a constant price-demand relationship might not reflect the true costs of the system resulting in inefficiencies. Demand in many industries benefit from quantity discounts, or bulk purchases for a lower price. However, constant price-demand prices, such as convex hull pricing, can never reflect the cost savings due to higher generation production. This paper strives to create a pricing algorithm that reflects market efficiency, and therefore we do not limit the method to one which will produce stable prices or prices that increase with demand. The following two sections discuss discriminatory pricing and assumptions made in the formulation.

A. Uplift Allocation: Ramsey-Boiteux Pricing

Without appropriate allocation, even distribution of uplift payments can provide misleading signals for investment. The DPA aims to allocate uplift judiciously, justifying the payments and charges with a scheme defined by Ramsey in 1927 [7]. Ramsey proposed a pricing method that allocates costs of taxes or fixed costs to consumers based on their willingness to pay. This is often called the inverse elasticity pricing rule, and requires transparency on cost and value functions. Ramsey's result shows in the presence of fixed costs, the efficient result can be discriminatory pricing in proportion to demand elasticity, i.e. the language of the Federal Power Act would define it as not unduly discriminatory since it is efficiency enhancing.

The result was extended by Boiteux in 1956 for electricity markets, differentiating a public and private price [8]. Ramsey-Boiteux pricing separates the single Ramsey price into a public price charged to all demand and a private discriminatory price that is different for each consumer based on that customer's elasticity. Demand that is more elastic, with a lower marginal value, will pay less of the fixed cost. Demand that is more inelastic, with a higher marginal value, will pay more of the fixed cost. Described in detail in Section V, the DPA introduces uplift payments and charges for both demand and generation. In order to maintain non-confiscation for both parties, uplift is distributed according to the bids and offers placed in the market.

B. Assumptions

There are several assumptions made in the model and algorithm. The first is that demand is not infinitely valued. We assume that demand bids their true value into the market; although it is possible that the bid is large (>> supply offer), it is not infinite. Many markets today allow for price responsive demand, though penetration is low. Second, the DPA model is intentionally simple in order to examine the new mechanism without introducing complications of a network. The addition of an electric network and other generation constraints will be added in subsequent analysis.

Finally, since following the efficient dispatch along with the LMP may cause participants to forego additional profits, lost opportunity cost (LOC) payments or administrative penalties are needed. Participants can be paid a LOC as an incentive to stay on the efficient dispatch, or they can be assessed a penalty for deviating from the efficient dispatch. While both penalties and LOCs may achieve the same outcome for generators – incentive to stay at dispatch level – LOCs may result in revenue inadequacy. If there is not enough market surplus to meet the LOC payment, then it is revenue inadequate, meaning there is a shortfall. The market surplus will determine if there are funds available to pay additional operating costs. Since the payment is not presently accounted for in the clearing mechanism, funding for LOC payments cannot be excluded from a new pricing proposal.

Administrative penalties may be used in place of LOCs for not following dispatch instructions. Such penalties should be clearly stated in the tariffs, which a market participant agrees to when entering the market. If the transparency of the penalty is a concern, it can be calculated just as the LOC and sent to market participants along with dispatch instructions. In transactions costs economics, contracts with rewards and penalties can result in an efficient contract. The implicit assumption that is now explicit is that a unit not selected cannot self-schedule or faces a penalty high enough to prevent self-dispatch.

IV. Literature Review

The literature on non-convex pricing in electricity markets can broadly be divided into proposals that advocate for a single market clearing price, and those that impose two- or multi-part pricing. Markets today use multi-part pricing; a clearing price and side payments, including uplift payments. The difficulty in side-payments is determining how they should be allocated. Most schemes do not include specific allocation instructions, leading to inefficiencies such as the historical examples mentioned in Section II. Alternatively, a single market clearing price is one known by all participants in and out of the market. The price must be high enough to cover all costs, and will therefore be non-confiscatory. Given the difficulties in non-convex pricing, it is important to evaluate potential implications of new pricing methods.

O'Neill et al. provided a foundational model for two-part pricing of electricity that supports the optimal schedule [9]. The locational and public price is determined from the dual variable of the node balance constraint in a linear model of the UC problem by fixing the binary variables. The second part of the price is the make-whole payment, which is the cost to cover a generator's fixed costs. As mentioned in Section II, markets today use an approach similar to [9], with exceptions for subsets of generation, such as fast-start generators. Convex hull pricing, proposed in [10] and [11], minimizes total uplift by creating the convex hull of the supply curve so that costs are a non-decreasing convex function of load. Researchers in collaboration with the Midcontinent Independent System Operator have suggested solution techniques for the convex hull in [12], [13], and have implemented an Extended LMP (ELMP), which relaxes the binary between zero and one [14]. Bjørndal and Jörnsten modify the prices from [9] to create less volatile prices and uplift charges [6]. Using the same example modified in [10], they show increasing stability of average prices compared to [9]. Other models attempt to internalize uplift prices with zero-sum transfers between generators, including a "general uplift approach" using a quadratic objective in [15] and [16], and a "minimum zero-sum uplift" model that ensures all generators break even [5]. The method in [5] ensures profitable generators do not to increase their profits by transferring additional payments to unprofitable generators. Van Vyve proposes a non-confiscatory pricing method with separate and allocated uplift payments in [17]. Finally, three methods attempt to create a single price that can cover both marginal and fixed costs. In [18] they use the solution technique of Lagrangian Relaxation to create a Semi-Lagrangian Relaxation, which relaxes node balance constraint and adds it to objective with a penalty price.

That price is found by iterating to obtain the same objective as the original MIP and raises the clearing price to cover any fixed costs. In [19], they use both the primal and dual constraints to increase the clearing price to provide non-negative profits to all generators. Although not explicitly proposed in the literature, average incremental cost (AIC) pricing can be used to create a single market clearing price. In a single period model, the method replaces the cost function with the average incremental cost of the generator at its optimal dispatch point, eliminating the binary variable and relaxing the minimum operating level to zero. Additional literature addressing non-convex pricing in electricity markets that does not directly suggest a new methodology can be found in [20], [21].

In a review article, Liberopoulos and Andrianesis analytically compare many non-convex pricing methods to determine the relative prices, payments and profits that result from each method [5]; they find no method dominates with respect to pricing criteria. In a similar vein to their comparison table, Table I shows a comparison of many of the methods described above. The columns show individual methodologies and the rows describe economic principles used to evaluate each method. These principles are the same as those described in Section III, principles that are essential for any pricing mechanism: maximizing market surplus, non-confiscation, revenue neutrality, and maintaining the optimal dispatch. Methods where uplift payments are determined outside of the model do not guarantee revenue adequacy (and therefore neutrality), since there might not be enough surplus from demand to pay the side-payment. All methods account for non-confiscation of supply offers; however, not all explicitly account for non-confiscation of demand bids. The third row indicates whether or not the demand side was *explicitly* incorporated or if non-confiscation of demand is enforced through the pricing mechanism by itself. Transparency is designated in the fourth row. Any mechanism which includes uplift has a discriminatory and private payment; unless the payments are made public, the pricing scheme cannot be considered wholly transparent. The DPA can be conditioned to provide either a single or two-part price, making transparency dependent on conditioning. The penultimate row describes the uplift present in the problem, whether it is allocated internally or determined after the pricing run (ex-post), or zero for single part pricing. It can also be noted that all but the second method incorporates fixed costs in some form into the price, whereas the price resulting from [9] will be a unit's marginal cost. The last row defines a mathematical category for the pricing problem proposed, with one category defined loosely as "LP+". This category is meant to encompass math programs that can be linear, but are nontrivial to determine at each implementation. The remaining categories are linear (LP), convex hull (CH), mixed integer program (MIP), and non-linear program (NLP). The only method without a direct citation is AIC, or average incremental cost pricing, which was described in the previous paragraph. The table does not necessarily suggest a single dominant method, but can be used as an evaluation tool for the pricing schemes. A system operator can evaluate the most pressing concerns for the region: price transparency, investment compatibility, incenting staying on dispatch, etc.

Methods for non-convex pricing must recover costs through an allocation system, which is often distributed evenly by levying a fixed \$/MWh fee to all loads. Few of the two-part pricing methods explicitly describe how uplift costs will be allocated, and there is little allocation literature focused on electricity. A general discussion of the theory and applications of cost allocation to many industries can be found in a series of essays edited by Young [22]. Electric market literature on cost allocation is mainly focused on transmission investments, such as a comparison of methods for cost allocation of transmission lines [23], [24]. As discussed in

Section III.A, we use Ramsey-Boiteux pricing to determine cost allocation through demand elasticity.

TABLE I
COMPARISON OF NON-CONVEX PRICING METHODS

	Two-Part Pricing							Single Price		
	Schweppe [25]	O'Neill [9]	Gribik [10], [11]	ELMP [14]	Bjørndal [6]	Galiana [15], [16]	DPA [4]	AIC	Araoz [18]	Ruiz [19]
Maximize market surplus	Y	Y	Y	Y	Y	Y	Y	Y	Y	N
Revenue neutral	Y	N	N	N	N	Y	Y	Y	Y	Y
Includes demand side	Y	N	N	N	N	Y	Y	Y	Y	Y
Maintain optimal dispatch	Y	Y	Y	Y	Y	Y	Y	Y	Y	N
Transparency	Y	N	N	N	N	N	Y/N	Y	Y	Y
Uplifts	Ex-post	Ex-post	Ex-post	Ex-post	Ex-post	Internal	Internal	None	None	None
Pricing problem type	LP	LP	CH	LP	LP+	NLP	LP	LP	LP+	MIP*

* Combination scheduling and pricing run, linearized MINLP; all other methods are post-UC pricing runs

V. DUAL PRICING ALGORITHM

The single-period model is described in minimal detail in [4]. Below we explain the derivation of the DPA constraints using a multi-period model, showing the canonical unit commitment problem and dual for reference.

A. Unit Commitment Model and Dual

The formulation in (1) is the canonical post-unit commitment problem. The unit commitment problem replaces (1f) and (1g) with $u_{it}, z_{it} \in \{0,1\}$ and is mixed integer program, whereas (1) is a linear program.

$$\max \sum_{t\in T}\left(\sum_{i\in D} b_{it}d_{it} - \sum_{i\in G}\left(c_{it}p_{it} + c_{it}^{OC}u_{it} + c_i^{SU}z_{it}\right)\right) \tag{1a}$$

$$\sum_{i\in D} d_{it} - \sum_{i\in G} p_{it} = 0 \qquad \forall t \in T \qquad \lambda_t \tag{1b}$$

$$p_i^{min}u_{it} \le p_{it} \le p_i^{max}u_{it} \qquad \forall i \in G, t \in T \qquad \beta_i^{max}, \beta_i^{min} \tag{1c}$$

$$u_{it} - u_{i,t-1} \le z_{it} \qquad \forall i \in G, t \in T \qquad \delta_{it}^{SU} \tag{1d}$$

$$0 \le d_{it} \le d_i^{max} \qquad \forall i \in D, t \in T \qquad \alpha_{it}^{max} \tag{1e}$$

$$u_{it} = u_{it}^{*} \qquad \forall i \in G, t \in T \qquad \delta_{it}^{u} \qquad (1f)$$

$$z_{it} = z_{it}^{*} \qquad \forall i \in G, t \in T \qquad \delta_{it}^{z} \qquad (1g)$$

The dual formulation is:

$$\min \sum_{t\in T}\left(\sum_{i\in D} d_i^{max}\alpha_{it}^{max} - \sum_{i\in G}(u_{it}^{*}\delta_{it}^{u} + z_{it}^{*}\delta_{it}^{z})\right) \qquad (2a)$$

$$\lambda_t + \alpha_{it}^{max} \geq b_{it} \qquad \forall i \in D, t \in T \qquad (2b)$$

$$-\lambda_t + \beta_{it}^{max} - \beta_{it}^{min} \geq -c_{it} \qquad \forall i \in G, t \in T \qquad (2c)$$

$$\delta_{it}^{SU} - \delta_{i,t+1}^{SU} + \delta_{it}^{u} - p_i^{max}\beta_{it}^{max} + p_i^{min}\beta_{it}^{min} = -c_{it}^{OC} \qquad \forall i \in G, t \in T \qquad (2d)$$

$$\delta_{it}^{z} - \delta_{it}^{SU} = -c_{it}^{SU} \qquad \forall i \in G, t \in T \qquad (2e)$$

$$\alpha_{it}^{max}, \beta_{it}^{max}, \beta_{it}^{min} \geq 0 \qquad \forall i \in D \cup G, t \in T \qquad (2f)$$

Using the dual formulation, we can formulate the dual pricing algorithm using the economic principles discussed in Section II. From strong duality of the primal and dual linear program in (1) and (2) respectively, we claim,

$$\sum_{t\in T}\left(\sum_{i\in D} b_{it}d_{it} - \sum_{i\in G}\left(c_{it}p_{it} + c_{it}^{OC}u_{it} + c_i^{SU}z_{it}\right)\right) = \sum_{t\in T}\left(\sum_{i\in D} d_i^{max}\alpha_{it}^{max} - \sum_{i\in G}(u_{it}^{*}\delta_{it}^{u} + z_{it}^{*}\delta_{it}^{z})\right). \qquad (2g)$$

For generators, the non-confiscation conditions for supply in (2c) and (2d) must be modified to reflect the startup decision. For $i \in G$, from (2c) and complementary slackness,

$$\left(-\lambda_t^{*} + \beta_{it}^{max*} - \beta_{it}^{min*} + c_{it}\right)p_{it}^{*} = 0 \qquad i \in G, t \in T \qquad (2h)$$

If $u_i^{*} = 1$, then from (1c) and complementary slackness,

$$(p_{it}^{*} - p_i^{max})\beta_{it}^{max*} = 0 \qquad i \in G | u_i^{*} = 1, t \in T \qquad (2i)$$

$$(p_{it}^{*} - p_i^{min})\beta_{it}^{min*} = 0 \qquad i \in G | u_i^{*} = 1, t \in T \qquad (2j)$$

Substituting this condition with (2i) and (2j), a one period model (excluding δ_{it}^{SU}) produces the classic economic result that profits are revenue less costs, or

$$\delta_{it}^{u*} = p_{it}^{*}(\lambda_t^{*} - c_{it}) - c_{it}^{OC} \qquad i \in G, T \in \{1\}. \qquad (2k)$$

That is, δ_{it}^{u*} is the LMP payment less the marginal and fixed costs incurred. Unfortunately, there is no guarantee that δ_{it}^{u*} is non-negative, that is, non-confiscatory. In the multi-period case, we sum together the δ_{it}^{u*} constraints in (2d) for all periods to create (2l) and reorder (2e) to obtain constraint (2m). The sum of (2l) and (2m) then define a total linear profit function for the operating period. Both (2i) and (2j) are again substituted into (2d), producing a total profit function for a multi-period model in (2n), where T_1 refers to the period in which the generator starts up, T_r refers to the total run period (all startups to shutdowns), T as a subscript refers to any period in the total time horizon, and $\delta_{iT_r}^{u*}$ sums all δ_{it}^{u*} for a generator's total run time, i.e., $\delta_{iT_r}^{u*} = \sum_{t\in T_r} \delta_{it}^{u*}$.

$$\delta_{iT_r}^{u*} = \sum\nolimits_{t\in T_r} \left(p_{it}^*(\lambda_t^* - c_{it}) - c_{it}^{OC}\right) - \delta_{iT_1}^{SU} \qquad i \in G \qquad (2l)$$

$$\delta_{iT_1}^{z*} = \delta_{iT_1}^{SU} - c_i^{SU} \qquad i \in G \qquad (2m)$$

$$\begin{aligned} \Pi_i &= \delta_{iT_r}^{u*} + \delta_{iT_1}^{z*} \qquad i \in G \qquad (2n) \\ &= \sum\nolimits_{t\in T_r} \left(p_{it}^*(\lambda_t^* - c_{it}) - c_{it}^{OC}\right) - c_i^{SU} \end{aligned}$$

Similar to the single period model, (2n) provides the classic economic result that the total profit under LMP pricing for generation is the payment received less the variable and fixed costs.

Due to degeneracy of the upper and lower bounds of (1c) in the primal problem when $u_{it} = 0$, we can apply an ε-perturbation method in order to determine the value of δ_{it}^{u*} at this solution. From (2c) and the resulting analysis, we have

$$\beta_{it}^{max*} \geq \lambda_t^* - c_{it} \qquad i \in G\ |u_{iT}^* = 0, t \in T. \qquad (2o)$$

Substituting into the startup condition of (2d), and using the same modifications as described in (2n), we obtain

$$\Pi_i = \sum\nolimits_{t\in T_r} \left(p_{it}^{max}(\lambda_t^* - c_{it}) - c_{it}^{OC}\right) - c_i^{SU} \qquad i \in G \qquad (2p)$$

We have the following four potential outcomes for u_{it}^* that demonstrate the need for make-whole payment and penalties.

If $u_{iT}^* = 1$ and $\delta_{iT_r}^{u*} + \delta_{iT_1}^{z*} < 0$, then a make-whole payment, $-\left(\delta_{iT_r}^{u*} + \delta_{iT_1}^{z*}\right)$, in addition to the LMP payment, which does not cover the offered cost, is needed to avoid confiscation, as we demonstrate below.

If $u_{iT}^{*}=1$ and $\delta_{iT_r}^{u*}+\delta_{iT_1}^{z*}\geq 0$, then a make-whole payment is not needed to avoid confiscation. The generator will either profit or break even from its LMP payment.

If $u_{iT}^{*}=0$ and $\delta_{iT_r}^{u*}+\delta_{iT_1}^{z*}>0$, then the LMP and a penalty or an LOC payment is needed to avoid the wrong price signal and 'decentralize' the market. The LMP with penalty provides enough disincentive to price-chasing and self-scheduling behavior. Without sufficiently large penalties, the generator would be indifferent to self-dispatching.

If $u_{iT}^{*}=0$ and $\delta_{iT_r}^{u*}+\delta_{iT_1}^{z*}\leq 0$, then the LMP sends the correct price signal. The generator would not profit from an LMP payment.

A two-part settlement (for example, LMP and make-whole payment) or a three-part settlement (for example, LMP, make-whole payment and penalties) are options for non-convex market clearing. We will assume here that the penalty for self-scheduling is high enough to prevent inefficient dispatch and the dispatch signal is a quantity signal.

Without make-whole payments, the results from (1) can be confiscatory. We add constraints to the equilibrium conditions on the market-clearing quantity to make the reallocation non-confiscatory. Therefore, the dual pricing algorithm enables a reallocation of profits for make-whole payments without altering the total market surplus, while maintaining revenue neutrality. Consequently the λ_t^{DPA} is no longer necessarily the LMP, but generally equals the average incremental cost, which is a better price signal.

The DPA scheme guarantees non-confiscation of demand bids. We substitute $\left(u_{it}^{pd}-u_{it}^{cd}\right)$ for α_{it}^{max*} and λ_t^{DPA} for λ_t^{*} in the complementary slackness condition from (2b), i.e.

$$d_{it}^{*}\left(b_{it}-\lambda_t^{DPA}+u_{it}^{pd}-u_{it}^{cd}\right)=0 \qquad \forall i\in D \qquad (2q)$$

This relationship is then summed over the commitment period to account for all payments and charges. The net value, $d_{it}^{*}b_{it}-d_{it}^{*}\lambda_t^{*}$, for $d_{it}^{*}>0$ can be defined as

$$\Psi_i = \sum\nolimits_{t \in T_r} d_{it}^* \left(b_{it} - \lambda_t^{DPA} + u_{it}^{pd} - u_{it}^{cd} \right) \qquad \forall i \in D \qquad (2r)$$

The net value must be nonnegative to ensure non-confiscation, enforced by

$$\Psi_i \geq 0 \qquad \forall i \in D \qquad (2s)$$

Since the market surplus is positive, the uplift payments and charges simply reallocate market surplus. Uplift payments and charges are participant specific, avoiding confiscation of any one participant. Without discriminatory uplift pricing, make-whole payments recovered uniformly across demand could result in confiscation. On the contrary, it is also unduly discriminatory to allocate uplift costs not based on cost causation principles.

For demand bids $i \in D$ not selected (i.e., $d_{iT}^* = 0, \alpha_{it}^{max*} = 0$), the net profit is zero, $\Psi_i = 0$. This is true for any feasible solution to the non-convex market model in (1). Substituting λ_t^{DPA} for λ_t^* in (2b), we obtain

$$\lambda_t^{DPA} \geq b_{it} \qquad i \in D \mid d_i^* = 0, t \in T. \qquad (2t)$$

This implies new λ_t^{DPA} should not low enough to entice an out-of-market bid to consume. In other words, unserved load will prefer not to take recourse actions that will lower the market surplus.

The DPA scheme guarantees non-confiscation of generator supply offers. We demonstrate above that the profit as defined in (2k) and (2n) can be negative. To ensure non-confiscation in the DPA, we introduce an uplift payment, u_{it}^p, and uplift charge, u_{it}^c, that can be impose for each market participant. We can redefine the profit condition in (2n) with non-confiscation of the profits Π_i as

$$\Pi_i = \sum\nolimits_{t \in T_r} \left(p_{it}^* \left(\lambda_t^* - c_{it} + u_{it}^p - u_{it}^c \right) - c_i^{OC} \right) - c_{it_1}^{SU} \qquad i \in G \quad (2u)$$

where we substitute λ_t^{DPA} for λ_t^* and introduce the uplift quantity $p_{it}^*\left(u_{it}^p - u_{it}^c\right)$ in each period. For supply, the profit or non-confiscation condition now is

$$\Pi_i \geq 0 \qquad i \in G \mid p_i^* > 0 \qquad (2v)$$

Since the DPA scheme is run post unit commitment, the criteria to maintain market surplus is already satisfied for the optimal solution to $(p_{it}^*, d_{it}^*, u_{it}^*, z_{it}^*)$; therefore constraint (2w) is enforced but redundant in the formulation.

$$\sum_{i \in G} \Pi_i + \sum_{i \in D} \Psi_i = MS^* \qquad (2w)$$

To ensure revenue adequacy, we balance the uplift payments and uplift charges through the following revenue neutrality condition

$$\sum_{t \in T}\left[\sum_{i \in D^+} d_i^*\left(u_i^p - u_i^c\right) + \sum_{i \in G^+} p_i^*\left(u_i^p - u_i^c\right)\right] = 0 \qquad (2x)$$

B. Conditioning

There are many possible prices that can result from the DPA. The DPA method allows the price to be adjusted based on the preference of the operator and market participants. Specific allocation criteria can be embedded in the model and produce conditioning such as perceived equity or increased transparency. If the region wishes to keep prices close to the dispatch LMP, the DPA maintain close prices and additionally allocate the uplift. If the market prefers a single market clearing price with no uplift payments, the algorithm will determine a single price. By providing tuning capabilities, we acknowledge that factors outside of the mathematical formulation or economic theory can drive decision making.

In order to condition the price, many small adjustments can be made depending on preference. We offer two options to ‘tune’ the price. Both condition the LMP by keeping the new price, λ_t^{DPA}, close to the dispatch LMP, λ_t^*, with penalties for deviations. To minimize the relative deviation, we constructed (3c). The second option in (2y) minimizes the absolute deviation across time.

$$\lambda_t^{\mathrm{DPA}} - \lambda_t^* - \lambda_t^{up} + \lambda_t^{dn} = 0 \qquad \forall t \in T \qquad (2y)$$

If there was concern about price spikes, we also may want to condition the uplift payments. Many possibilities can be considered. One possibility is to limit the maximum allowable payment and charge by the constraints listed in (2z). This may result in insufficient cost allocation.

$$\begin{aligned} u_{it}^p \le u_i^{p\max}, u_{it}^c \le u_i^{c\max} & \qquad \forall i \in G^+, t \in T \\ u_{it}^{pd} \le u_i^{pd\max}, u_{it}^{cd} \le u_i^{cd\max} & \qquad \forall i \in D^+, t \in T \end{aligned} \qquad (2z)$$

With potentially higher prices, higher penalties can be calculated based on λ_t^{DPA}. With sufficiently large penalties of at least $\sum_{t\in T}\left(p_{it}^{\max}(\lambda_t^* - c_{it}) - c_{it}^{OC}\right) - c_i^{SU}$, the generator becomes indifferent to self-dispatching. Just like a lost opportunity cost payment, these penalties can be sent along with the price and quantity dispatch signal.

C. Dual Pricing Algorithm (DPA) Formulation

We now formulate the DPA model in (3) using the modifications of the dual problem described in Section V.B.

$$\min \sum_{t\in T}\left[\sum_{i\in D^+} d_{it}^* u_{it}^{pd} + \sum_{i\in G^+} p_{it}^* u_{it}^p + c^{up}\lambda_t^{up} + c^{dn}\lambda_t^{dn}\right] \qquad (3a)$$

$$\sum_{t\in T}\left[\sum_{i\in D^+} d_{it}^*\left(u_{it}^p - u_{it}^c\right) + \sum_{i\in G^+} p_{it}^*\left(u_{it}^p - u_{it}^c\right)\right] = 0 \qquad (3b)$$

$$(\lambda_t^{DPA} - \lambda_t^*)/\lambda_t^* - \lambda_t^{up} + \lambda_t^{dn} = 0 \qquad (3c)$$

$$\Psi_i = \sum_{t\in T_r} d_{it}^*\left(b_{it} - \lambda_t^{DPA} + u_{it}^{pd} - u_{it}^{cd}\right) \qquad \forall i \in D^+ \qquad (3d)$$

$$\Pi_i = \sum_{t\in T_r}\left(p_{it}^*\left(\lambda_t^{DPA} - c_{it} + u_{it}^p - u_{it}^c\right) - u_{it}^* c_{it}^{OC}\right) - z_{iT_1}^* c_i^{SU} \qquad \forall i \in G^+ \qquad (3e)$$

$$\begin{aligned} \lambda_t^{DPA} \ge b_{it} & \qquad \forall i \in D^0, t \in T \qquad (3f) \\ \Psi_i \ge 0 & \qquad \forall i \in D^+ \qquad (3g) \\ \Pi_i \ge 0 & \qquad \forall i \in G^+ \qquad (3h) \\ u_{it}^p, u_{it}^c, u_{it}^{pd}, u_{it}^{cd} \ge 0 & \qquad \forall i \in D \cup G, t \in T \qquad (3i) \end{aligned}$$

Theorem 1. If there exists an optimal solution to the primal problem in (1), that is, the maximize market surplus problem, then it is a feasible solution to DPA.

Proof. A feasible solution to (1) is obtained with $d_{it}^* = p_{it}^* = z_{it}^* = 0$ and $MS = 0$. From the optimization of (1) and summing together (3b), (3d), and (3e), we have $MS^* = \sum_{i \in G} \Pi_i + \sum_{i \in D} \Psi_i \geq 0$.

From complementary slackness of (1c) with the binary fixed at its optimal value, there are three possible cases for the values of $p_{it}^*, \beta_{it}^{max*}$, and β_{it}^{min*}:

(1) if $p_{it}^* = p_i^{max}$, then $\beta_{it}^{max*} > 0$ and $\beta_{it}^{min*} = 0$;

(2) if $p_{it}^* = p_i^{min}$, then $\beta_{it}^{max*} = 0$ and $\beta_{it}^{min*} > 0$;

(3) if $p_{it}^* \in (p_i^{min}, p_i^{max})$, then $\beta_{it}^{max*} = \beta_{it}^{min*} = 0$.

Therefore $p_{it}^*(\beta_{it}^{max*} - \beta_{it}^{min*}) = p_{it}^{max}\beta_{it}^{max*} - p_{it}^{min}\beta_{it}^{min*}$. From complementary slackness of (2c), $p_{it}^*(\lambda_t^* - c_{it}) = p_{it}^*(\beta_{it}^{max*} - \beta_{it}^{min*})$. As shown in (2n), $\delta_{iT_r}^{u*} + \delta_{iT_1}^{z*}$ is the linear surplus of generator $i \in G$. From complementary slackness of (2b), $d_{it}^*(b_{it} - \lambda_t^*) = d_{it}^*\alpha_{it}^{max*}$, and $d_{it}^*\alpha_{it}^{max*} \geq 0$ since d_{it}^* and α_{it}^{max*} are both nonnegative.

We partition $i \in G$ into three sets $G' = \{i \in G: \delta_{iT_r}^{u*} + \delta_{iT_1}^{z*} \geq 0 \text{ and } u_{iT_r}^* = 1\}$, $G'' = \{i \in G: \delta_{iT_r}^{u*} + \delta_{iT_1}^{z*} < 0 \text{ and } u_{iT_r}^* = 1\}$, and $G''' = \{i \in G: u_{it}^* = 0\}$. $\Pi_i = 0$ for all $i \in G'''$.

$$\begin{aligned} MS^* &= \textstyle\sum_{i \in G} \Pi_i + \sum_{i \in D} \Psi_i \\ &= \textstyle\sum_{i \in G'} \Pi_i + \sum_{i \in G''} \Pi_i + \sum_{i \in D} \Psi_i \geq 0 \\ &= \textstyle\sum_{i \in G'} \Pi_i + \sum_{i \in D} \Psi_i \geq -\sum_{i \in G''} \Pi_i \end{aligned}$$

Let $\lambda_t^{\mathrm{DPA}} = \lambda_t^*$ and use the previously mentioned complementary slackness conditions to see that $\Pi_i = \delta_{iT_r}^{u*} + \delta_{iT_1}^{z*} + \sum_{t \in T}(p_{it}^* u_{it}^p - p_{it}^* u_{it}^c)$ and $\Psi_i = \sum_{t \in T} d_{it}^*\alpha_{it}^{max*} + d_{it}^* u_{it}^{pd} - d_{it}^* u_{it}^{cd}$.

Let $u_{it}^p = u_{it}^{pd} = 0$ on the LHS, $u_{it}^c = 0$ on the RHS, and substituting for Π_i and Ψ_i:

$$\sum_{i\in G'}\left(\delta^{u*}_{iT_r}+\delta^{z*}_{iT_1}-\sum_{t\in T}p^*_{it}u^c_{it}\right)+\sum_{i\in D,T}d^*_{it}\alpha^{max*}_{it}-d^*_{it}u^{cd}_{it}$$

$$\geq -\sum_{i\in G''}\delta^{u*}_{iT_r}+\delta^{z*}_{iT_1}+\sum_{t\in T}p^*_{it}u^p_{it}$$

Then we can select payments and charges. For all $i\in G''$, let $\sum_{t\in T}p^*_{it}u^p_{it}=-\left(\delta^{u*}_{iT_r}+\delta^{z*}_{iT_1}\right)>0$. This satisfies (3f) for $i\in G''$. For all $i\in G'$ select u^c_{it} and u^{cd}_{it} such that,

$$\sum_{t\in T}\sum_{i\in G'}p^*_i u^c_i+\sum_{i\in D}d^*_i u^{cd}_i=\sum_{t\in T}\sum_{i\in G''}p^*_{it}u^p_{it},$$

$\sum_{t\in T}p^*_{it}u^c_{it}\leq \delta^{u*}_{iT_r}+\delta^{z*}_{iT_1}$ for $i\in G'$, and

$\sum_{t\in T}d^*_{it}u^{cd}_{it}\leq \sum_{t\in T}d^*_{it}\alpha^{max}_{it}$ for $i\in D$.

The selection criteria are equivalent to (3b), (3g) and (3h).

From strong duality of (1) and (2), we know that $MS^*=\sum_{i\in G}\delta^{u*}_{iT_r}+\delta^{z*}_{iT_1}+\sum_{i\in D,t\in T}d^*_{it}\alpha^{max*}_{it}\geq 0$, and we also have $\sum_{i\in G'}\delta^{u*}_{iT_r}+\delta^{z*}_{iT_1}+\sum_{i\in D,t\in T}d^*_{it}\alpha^{max*}_{it}\geq -\sum_{i\in G''}\delta^{u*}_{iT_r}+\delta^{z*}_{iT_1}$. Therefore, the generators $i\in G'$ and demands $i\in D$ have enough linear surplus to satisfy (3g) and (3h). Constraints (3b), (3d), (3e), and (3i) are satisfied by the construction.

For $d^*_{iT}=0$, complementary slackness requires $\lambda^*_t\geq b_{it}$, so (3f) is satisfied and the DPA has a feasible solution. ■

VI. RESULTS

We explore several examples to illustrate the capability and flexibility of the DPA. The first two examples show a one period model, demonstrating a payment and charge between demand and a comparison to popular examples in the literature. The next examples are multi-period models.

A. Single bus, single time period examples

One single period example can be found in [4], showing the resulting price increases just high enough to cover the fixed costs of both generators with an uplift payment and charge levied on the customers. Using data from a MISO sample problem [27], we can look at a range of demand levels for a single period problem. The generator costs are in Table II, and the single demand has a value of \$100/MWh. Fig. 1 shows the clearing price for three different pricing methods for demand from 0 MW – 350 MW. The dispatch LMP (λ^*) spikes when moving from the cheaper generators to the more expensive (A to C), since it must turn on the most expensive generator (D) to match demand. The ELMP is monotonically non-decreasing with demand, forming steps when the expensive generator is needed. Similar to the dispatch LMP, the DPA price also spikes when the expensive generator is dispatched. The prices then decrease, showing quantity discounts as the generator reaches it maximum. There are no uplift payments needed with the DPA, while there are payments required from the LMP and ELMP.

TABLE II
GENERATOR COSTS

Gen	Marginal cost (\$/MWh)	Startup cost (\$)	P^{min} (MW)	P^{max} (MW)
A	50	500	20	100
B	52	500	20	100
C	55	500	20	100
D	65	40	5	50

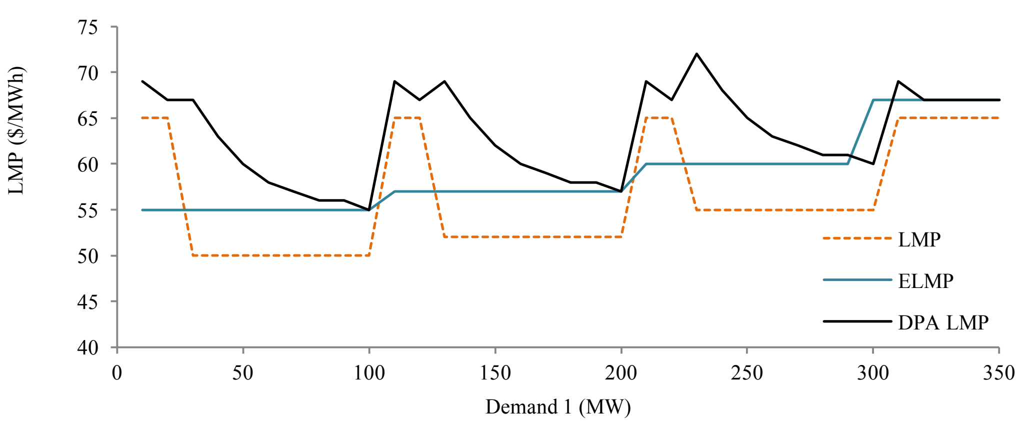

Fig. 1 Increasing demand with three different pricing methods

We simulated other small test examples with similar results. A benchmark example created by Scarf in [28] has been used to demonstrate the versatility of pricing methods. The DPA is compared with a traditional LMP and uplift in Fig. 2. The figure shows the changes in price as demand quantity increases. The prices and resulting uplift payments are shown with blue solid and dashed lines, while the DPA prices and uplift are shown in black and orange. There are no uplift payments made at any demand level, and prices oscillate between \$6/MWh and \$7/MWh, the latter being the price of the generator with a high marginal cost and no startup cost.

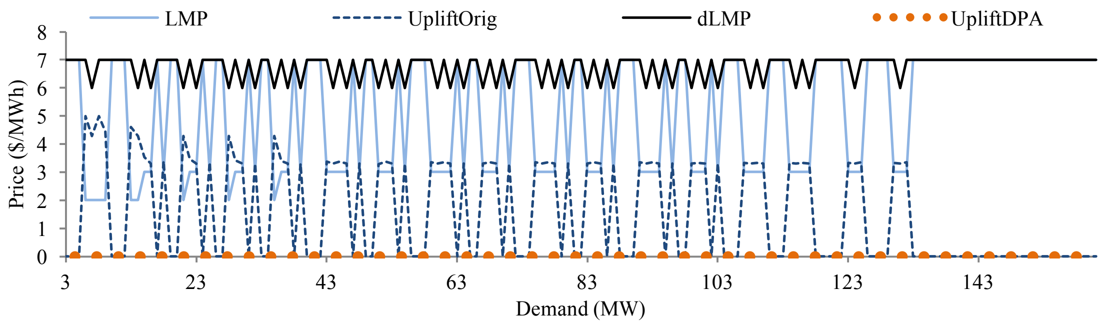

Fig. 2 Prices resulting from the modified Scarf example compared with the traditional method of determining prices and uplift payments

B. Multi-period Examples

The single period examples demonstrate the prices across varying demand levels, while the following multi-period examples focus on the prices and payments across time. The generator characteristics are found in Table III, and the demand value and quantity, and reserve data are found in Table IV.

TABLE III
GENERATOR DATA

Gen	Marginal cost (\$/MWh)	Startup cost (\$)	No Load cost (\$/h)	P^{min} (MW)	P^{max} (MW)
A	30	900	100	200	1200
B	50	600	100	50	80

TABLE IV
HOURLY DATA

	1	2	3	4	5	6	7	8
Demand 1, Value \$200/MWh	510	528	546	573	582	588	594	564

Demand 2, Value $80/MWh	340	352	364	382	388	392	396	376
Reserve (MW)	85	20	20	20	20	20	20	20

The resulting prices are in Table V, showing the difference between the two types of DPA conditioning and the traditional LMP, λ_t^*. In the first conditioned DPA price, λ_t^{DPA}, the penalties are imposed in every period, while $\lambda_t^{\mathrm{DPA}\prime}$ imposes a single penalty on all periods. The impact is uniform prices for all periods compared to a higher price in a single period. Another notable impact is on uplift payments. The dispatch LMP imposes a $1700 uplift payment on the system, while both conditioned DPA prices incur no uplift. This is an example where a the DPA produces a single market clearing price, and there is no need to follow uplift allocation guidelines.

TABLE V
PRICES & PAYMENTS

	1	2	3	4	5	6	7	8	Uplift ($)
λ_t^*	30	30	30	30	30	30	30	30	1700
λ_t^{DPA}	30.23	30.23	30.23	30.23	30.23	30.23	30.23	30.23	0
$\lambda_t^{\mathrm{DPA}\prime}$	30	30	30	30	30	30	31.72	30	0

1) Increased demand

The following example uses the same generator data shown in Table III, increased demand shown in Table VI, and higher values of c^{up} and c^{dn}. The resulting prices from the DPA match the dispatch LMP, as shown in Table VII. Necessarily, the total uplift payments are the same; however, the DPA allocates the uplift to the last period with a payment of $46/MWh to Generator B provided by a $4/MWh charge to Demand 2. If maintaining prices similar to the dispatch LMP is preferable to a system operator, this example shows conditioning the penalty factor can produce DPA prices that are identical to dispatch prices while allocating dispatch to particular participants.

TABLE VI
HOURLY DATA

	1	2	3	4	5	6	7	8
Demand 1, Value $200/MWh	600	608	626	653	662	668	674	644
Demand 2, Value $80/MWh	510	528	546	573	582	588	594	564

TABLE VII
PRICES & PAYMENTS

	1	2	3	4	5	6	7	8	Uplift ($)
λ_t^*	30	30	30	30	30	50	80	30	2300
λ_t^{DPA}	30	30	30	30	30	50	80	30*	2300

*uplift allocated to Gen. B and Dem. 2

2) *RTS test case*

Finally, we examine the generation from a modified single zone RTS96 test case [29]. The generator characteristics and load data are found in Table VIII and Fig. 2. All generators were located at a single node with 24 hourly simulations. The resulting generator profits and demand value are in Table IX. As expected, the total social welfare remains the same between the two simulations. In order to reduce uplift and provide proper incentives for investment, there is a transfer of surplus between consumers and producers. With zero uplift, the price provides a transparent indicator for investment; it allows investors to evaluate if their unit could enter the dispatch. While the only guaranteed method of analysis for market entry would involve rerunning the dispatch with the potential unit, the transparency of the DPA price sends a more efficient signal than a marginal pricing method alone.

TABLE VIII
GENERATOR DATA

Gen	Quantity	P^{max} (MW)	P^{min} (MW)	$c^{startup}$ ($)	$c^{marginal}$ ($/MWh)	c^{noload} ($/h)
Oil/CT	4	20	15.8	76	163	1139
Coal/Steam	4	76	15.2	1061	19.64	131
Oil/Steam	3	100	25	4754	75.64	840
Oil/Steam	3	197	68.95	6510	74.75	1160
Oil/Steam	5	12	2.4	571	94.74	73
Coal/Steam	2	155	54.25	1696	15.46	253
Nuclear	2	400	100	2400	5.46	215

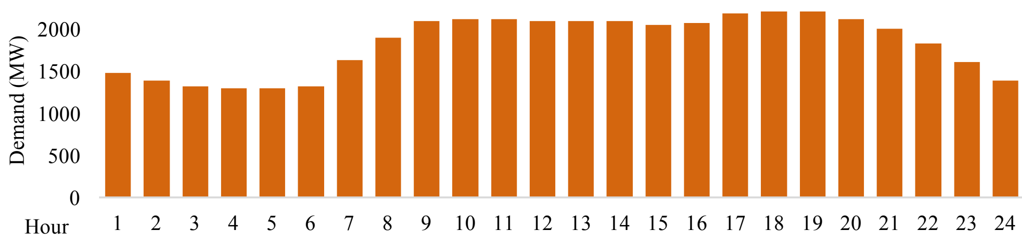

Fig. 3 Hourly demand for the modified RTS example

TABLE IX
SURPLUS & PAYMENTS

	Traditional LMP	DPA
Generator Profits	$2,244,014	$4,765,784
Consumer Value	$5,233,475	$2,711,704
Uplift	$10,768	$0

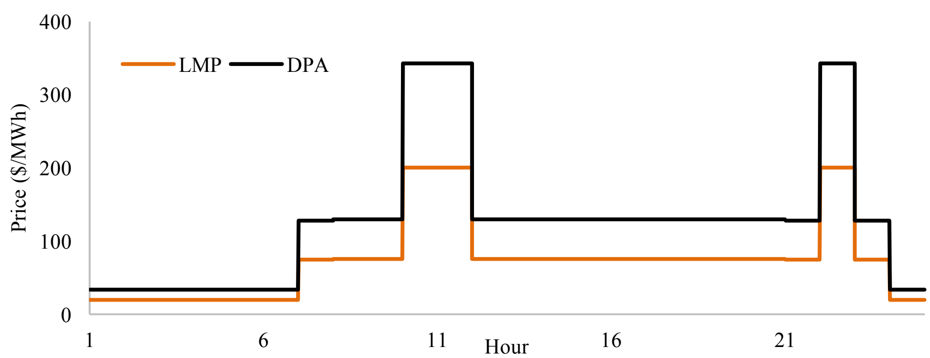

Fig. 4 Hourly price comparison

VII. Discussion

The examples in the previous section show the prices and costs that can result from the DPA. While not guaranteed to always occur, there are several common trends in the examples. DPA prices tend to be higher than the traditional LMP and the ELMP. This is not surprising due to the incorporation of fixed costs, and low or no uplift payments. Higher prices should not be perceived as positive or negative; however, when there are no private side payments, there is increased transparency in the market. Additionally, compared to pricing mechanisms that are non-decreasing (like convex hull), prices are more volatile. Due to fixed costs, DPA prices are closer to the average incremental cost of delivering power, which is a decreasing function with respect to demand for each generator. As discussed in Section III, volatility should not be considered an objectionable trait, rather one that can reveal the true value of producing power.

Pricing mechanisms should produce efficient prices, ones that support the optimal schedule. In combination with deviation penalties, the DPA prices support the optimal schedule and recover all parts of both generation bids and demand offers. The prices also signal points of entry into the market. With low or no uplift payments, new entrants can better evaluate if their incremental costs are below the clearing price. While this is not a guaranteed point of entry, it provides more market information than the traditional LMP. When side payments are needed, the algorithm allocates them to both supply and demand in particular periods. The endogenous allocation ensures that demand does not pay more than its offer and supply is made whole. In a one-sided market (where demand in inelastic), demand will pay any price and revenue adequacy is guaranteed. Even in markets today there are elastic bids, a number that is likely to increase as markets change in response to the shifting resource mix.

VIII. Conclusion

Spot prices should provide proper incentives for both operations and investment. Electricity is unlike other commodities due to the fixed costs necessarily incurred during operation and the need to physically balance supply and demand. Due to these non-convexities, it is difficult to determine the ‘right’ price for electricity. Methods suggested in the literature often consider only one aspect of pricing, or are contingent on inelastic demand. In this paper we propose the Dual Pricing Algorithm, which brings together many principles surrounding pricing mechanisms: maximizing market surplus, revenue neutrality, non-confiscation, transparency, signals for market entry, and side payment allocation. The DPA is an ex post pricing scheme that upholds these principles and can be adapted to particular system operator needs. It is a linear program, making it computationally efficient, and can be incorporated into current ISO software. The approach is applied to multiple time horizons and can easily include additional operational constraints, e.g., reserve requirements. Further work can be done to incorporate these constraints and evaluate the algorithm on a large network model.

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