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DYNAMIC OLIGOPOLISTIC COMPETITION ON AN ELECTRIC POWER NETWORK WITH RAMPING COSTS AND JOINT SALES CONSTRAINTS

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Abstract. Most previous Cournot-Nash models of competition among electricity generators have assumed a static perspective, resulting in finite dimensional variational and quasi-variational inequality formulations. However, these models' system costs and constraints fail to capture the dynamic nature of power networks. In this paper we propose a more general and complete model of Cournot-Nash competition on power networks that accounts for these features by including (*i*) explicit intra-day dynamics that describe the market's evolution from one Generalized Cournot-Nash Equilibrium to another for a 24 hour planning horizon, (*ii*) ramping constraints and costs for changing the power output of generators, and (*iii*) joint constraints that include variables from other generating companies within the profit maximization problems for individual generators. These joint constraints yield a generalized Nash equilibrium problem which can be represented as a differential quasi-variational inequality (DQVI); such generalized Nash equlibrium problems can have multiple solutions. The resulting formulation poses computational challenges that can cause traditional algorithms for DVIs to fail. A restricted formulation is proposed that can be solved by an implicit fixed point algorithm. A numerical example is provided.

1. **Introduction.**

One of the major areas of application of complementarity and variational inequalities-based models of economic equilibria is electric power markets; more so since this economically crucial industry underwent a transition from tight regulation to intense competition subject to loose regulatory constraints. It is not our intention, nor does space permit, to list or discuss all the previous models.

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However, Daxhalet and Smeers [3], Day et al. [4], Hobbs [10], Hobbs and Helman [11], Metzler et al. [12], Neuhoff et al. [15] and Pang and Hobbs [18] summarize the relatively recent literature on complementarity and variational inequalities-based models of electric power equilibrium problems. Unlike other engineered systems, technology and cost information is widely available for power industries which facilitates modeling. At the same time, the unique characteristics of electricity transmission, such as Kirchhoff's power and voltage laws, present intriguing challenges to modelers and systems engineers. Kirchhoff's laws arise from the inherent characteristic of flow of power in an electric network, namely the energy balance (known as the first law) and voltage law (also known as the second law). Sethi et al. [21], on the other hand, study the problem of optimal staged purchase of electricity in time-sequential deregulated electricity markets using a stochastic dynamic programming framework.

Complementarity and variational inequalities-based models can serve several practical purposes. One is numerical simulation of the economic and pollution consequences of alternative government policies (such as pollution limits or taxes) and market designs (such as pricing policies for transmission or creation of capacity markets). Because of the efficiency of large-scale complementarity solvers, these models can incorporate significant technical detail, such as transmission capacity limits and fuel use efficiencies, and provide fine scaled results concerning, e.g., prices over space and time. Another use of these models is theoretical comparison of different market designs and policies using more general and abstract models. Because of the large amount of theory that has been developed concerning complementarity problems and variational inequalities (some of which is used in this paper), it is often possible to either show or disprove equivalence of alternative market designs, as well as existence and uniqueness properties. Of course, such results can also be obtained for more traditional economic equilibrium models that lack inequality constraints. However, the ability to consider inequalities makes these models especially suitable for power systems, since those systems have many inequalities in the form of transmission and generation capacity limits. A third use of complementarity/variational inquality models is market monitoring. For instance, these models can be used to construct a "competitive baseline" that calculates prices assuming that suppliers are "price takers" who do not manipulate prices. Deviations between baseline and actually observed prices might provide evidence of anticompetitive behavior. Models can also be used to simulate the exercise of market power. It is not possible to reliably predict precise modes of oligopolistic behavior (such as Cournot Nash, Cournot Bertrand, Stackelberg, and tacit collusion) or exact price outcomes for particular markets. This is because, for instance, demand elasticities are uncertain, or the degree of cooperation among suppliers depends on many unknown factors. However, such models can serve as "possibility proofs", indicating what anticompetitive problems might arise in real markets and should therefore be monitored.

Compelementarity and variational inequalities-based models are increasing getting traction in studying and computing equilibria in the context of supply chains, pricing-allocation problems in service industries, and city logistics. Zhang [25], Mookherjee [14] and Friesz et al. [7] and the references therein provide an overview of such emerging applications.

An important feature omitted in most complementarity-based models is the possibility that an electricity producer can recognize joint constraints wherein the possible solution space for one player is affected by the decisions of the other players. For instance, regulators might impose an upper bound upon the market share of the few largest producers in some markets, or upon the proportion of transmission capacity that is sold to such producers, as is the case for transmission capacity into the Netherlands. Perhaps the most important example would be a recognition by a generator that its sales and generation are limited by available transmission capacity less that capacity which is already taken up by sales and generation by other producers. When considering elaborate network topologies, very complex sets of constraints on sales and generations may arise.

Another important feature missing from all existing electric power equilibrium models is the consideration of ramping costs. There are some instances of ramping cost-based models for the monopolistic firm, namely by Wang and Shahidehpour [24], Shrestha et al. [22], and Tanaka [23] consider a decomposed model for optimal generation scheduling of a cost minimizing monopolistic power generator explicitly considering ramping costs. However, the market perspective is missing in that model. On the other hand, Shrestha et al. [22] consider a dynamic model for strategic use of ramping rates beyond elastic limits in a power producer's self-dispatch in a power market with exogenous price and demand. Tanaka [23] focuses on derivation of a pricing policy that achieves the optimal rate of a demand change by explicitly considering the ramping cost. All three of these papers do not consider the network topology or Kirchhoff's laws.

Oren and Ross [16], on the other hand, study a unique phenomenon resulting from the incompleteness of the electricity supply market whereby profitable gaming of ramp constraints can be possible through appropriate generators. Some electric power markets are structured in such a way as to allow bidders to specify constraints on ramp rates for increasing or decreasing power production. By taking data from the actual demand for electricity in California, in 2001, they show that a bidder could apply an excessively restrictive constraint to increase profits, and explore the cause by visualizing the feasible region from the linear program corresponding to the power auction. They also propose three penalty approaches to discourage such activity: one approach based on social cost differences caused by ramp constraints; and two approaches based on the duality theory of linear programming. The authors then apply the resulting model to the characteristics of the electricity supply industry in Spain. The results indicated that the penalties based on linear program sensitivity theory (PP1) have the advantage that they are easily computed from a single run of the auction, while penalties based on the difference between ramp constrained social cost and the social cost without a bidder's ramp constraints (PP2) require many optimizations to be run, but they perform better on social cost recovery. This unique study helps to overcome the intrinsic problem associated with the electricity supply market in that they are notorious for having multiple optimal solutions.

Meanwhile, nearly all existing power market equilibrium models fail to take a dynamic perspective. These models do not consider short-run constraints and costs associated with changing generators output. Large generators can take up to ten or more hours to ramp up to full output, yet power demands can change drastically from one hour to the next. In order to match those changes, quick-starting but costly combustion turbines are turned on and off. The result is that the highest price spikes

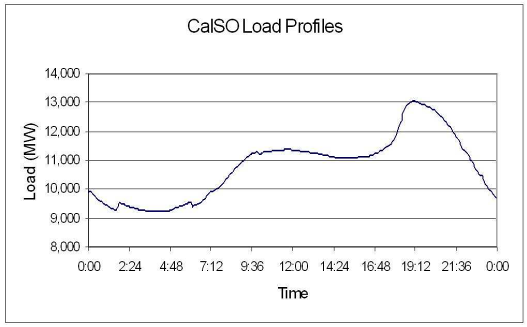

FIGURE 1. Load profile in CaISO market for a typical summer day in 2004

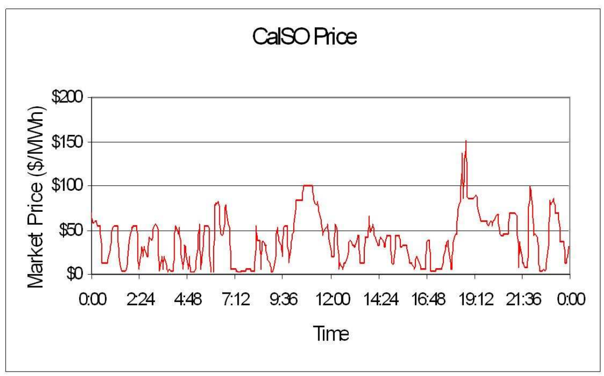

FIGURE 2. Corresponding spot market price in CaISO

often occurs not during the periods of high demand, but during the early morning or late afternoon when the demand is changing most quickly. Figures 1 and 2 show a typical daily load profile for the California Independent System Operator (ISO) (source: http://www.caiso.com) and the price spikes that result from ramping constraints respectively. These dynamic market phenomena can not be captured by the typical static formulation of equilibrium models that omits consideration of time.

We believe here that electric power systems display a so-called *moving-equilibrium* (see Friesz et al. [5]) wherein an equilibrium is enforced at each instant of time although state and control variables will generally fluctuate with time. These fluctuations with respect to time are exactly those needed to maintain the balance of behavioral and economic circumstances defining the equilibrium of interest.

In the current paper we have tried to tie all these missing components together. In doing so we consider a dynamic model of generalized Cournot-Nash (CN) equilibria of oligopolistic competition on an electric power network. The generalized CN equilibrium problem is an extension of the CN equilibrium problem, in which each player's strategy set depends on the rival players' strategies. The electric power network we consider here consists of spatially distributed markets, generating firms, and an ISO. The ISO is the principal agent for electricity transmission who receive fees from the generators. The ISO sets the transmission fees in order to efficiently clear the market for transmission capacity. We do not consider additional players in the markets such as fuel suppliers or arbitrageurs. We also do not consider refinements such as price function conjectures (Day et al. [4]). Nonetheless, our basic dynamic model of generalized CN equilibrium has some unique features that were overlooked in earlier papers; namely dynamics, ramping constraints and costs.

Including dynamic features can make market equilibrium models more useful for the purposes described earlier in the introduction. In particular, models that including ramping constraints can be used to identify situations in which transient market power can be exercised by generation owners who control units that provide critical ramping capability. Although these situations may be short in duration, they can involve significant costs to consumers. A second use of such models could be to analyze the value of increasing the flexibility of the generation system by providing more ramping capability, and the revenues that such capacity would receive. There has been controversy recently in some U.S. markets (PJM and California in particular) about whether there is enough rampable capacity, and if revenues are sufficient to compensate generators for the cost of providing it.

In the next section we put forward an optimal control-based formulation of the generators' profit maximization model explicitly considering ramping constraints and costs as well as joint constraints arising from market capacity constraints. We also show how such models give rise to a joint Nash game in a dynamic sense, and the equivalent dynamic quasi variational inequality formulation, solution of which is a solution of the Nash game. We also describe a variant of fixed point algorithm designed to solve this problem and illustrate the efficacy of such algorithm through a numerical example.

2. **The generating firms' problems.** The oligopolistic firms of interest are power generating firms embedded in a network economy. These firms compete in a CN oligopolistic game where each firm is attempting to maximize its own profits while adhering to both physical and regulatory constraints. The firms' profits are its revenues less costs. Instantaneous revenues are equal to regional sales in the market level times the corresponding nodal (market) prices, and the costs include generation costs, ramping costs and transmission fees, the latter paid to the ISO. Due to the inherent properties of an electrical power network, it is assumed that a firm located at some node of the network can supply energy to any other node (market) of the network.

The physical constraints arise from the physical characteristics of the actual power network as well as the physical attributes of the generating facilities. The flows of power in the network must obey Kirchhoff's power and voltage laws. The generating facilities have both lower and upper bounds on the level of power generation as well as lower and upper bounds on the rate at which the output of the generating units can be adjusted. Regulatory constraints may arise in the form of an upper limit on the total power provided to a particular market by all of the firms. The generating firms compete as price takers in the electric power markets with respect to the price of transmission services, consistent with assumptions made by previous complementarity-based models.

2.1. **Representation of Kirchhoff's laws.** Rather than explicitly considering Kirchhoff's laws, in this paper we will use the simpler representation of power

TABLE 1. Notation: Network Parameters

Parameters	
$\mathcal{N}$	set of nodes, excluding the hub, denoted H
$\mathcal{A}$	set of bi-directional arcs
H	hub node
$\mathcal{F}$	set of firms
$\mathcal{N}_f$	set of nodes where generating firm $f \in \mathcal{F}$ has power generators
$\mathcal{G}(i,f)$	set of power generating facilities owned by firm $f \in \mathcal{F}$ located at node $i \in \mathcal{N}_f$
T_a	transmission capacity on arc $a \in \mathcal{A}$
$PTDF_{ia}$	power transfer distribution factor for node i on arc a, describing the megawatt (MW) increase in flow on arc a resulting from 1 MW of power withdrawal at i and 1 MW of injection at H due to Kirchhoff's laws
CAP_j^f	generation capacity of plant $j \in \mathcal{G}(i,f)$ located at node $i \in \mathcal{N}_f$ owned by firm $f \in \mathcal{F}$
σ_i	regional sales cap at market $i \in \mathcal{N}$
R_j^{f+}	upper bound of ramping rate of the generator unit $j \in \mathcal{G}(i,f)$ located at node $i \in \mathcal{N}_f$ owned by firm $f \in \mathcal{F}$
R_j^{f-}	lower bound of ramping rate of the generator unit $j \in \mathcal{G}(i,f)$ located at node $i \in \mathcal{N}_f$ owned by firm $f \in \mathcal{F}$

transfer distribution factors (PTDFs) as derived, e.g., in the appendix A of Scheppe et al. [19]. These PTDFs can be computed offline and indicate what fraction of power will flow over each edge of the network when injecting 1 MW at node i and extracting 1 MW from node j. Because the system is linear, the PTDFs from j to i are simply the negative of the PTDFs from i to j. By considering an arbitrary hub node, we can compute the PTDFs for a 1 MW injection at the hub and a 1 MW extraction at node i for all nodes in the network. The PTDF for an injection at node i and an extraction at node j can be represented as the negative of the PTDF from the hub to node i plus the PTDF from the hub to node j. In addition to using the PTDF representation, flow balance is enforced for each generation firm; in each hour the sum of power produced from all its plants at all its nodes must equal the sum of all its sales at all node. Using this representation greatly simplifies the constraint set for each firm's problem.

2.2. **Notation.** We primarily employ the notation used in Miller et al. [13], augmented to handle temporal considerations. Time is denoted by the scalar $t \in \Re^1_+$, initial time by $t_0 \in \Re^1_+$, final time by $t_1 \in \Re^1_{++}$, with $t_0 < t_1$ so that $t \in [t_0, t_1] \subset \Re^1_+$. Other notation involved with our model is summarized in Tables 1 and 2.

2.3. The generating firms' problem formulation.

2.3.1. *Time Scale.* The regional demand for electricity varies across hours of a day. Over different days there is notable periodicity. For example, the demand for power at a specific hour of a day is very similar to the demands at the same hour on other days. In particular, we assume the period T to be 24 hours which is the planning

TABLE 2. Notation: State and Control Variables

Variables	
s_i^f	rate of sales of power (MW) at node $i \in \mathcal{N}$ by firm $f \in \mathcal{F}$
q_j^f	generation rate of the generator unit $j \in \mathcal{G}(i, f)$ located at node $i \in \mathcal{N}_f$ owned by firm $f \in \mathcal{F}$
y_i	amount of transmission in megawatts from hub H to node i
w_i	wheeling fee or price from hub H to node i (\$ / MW)
r_j^f	ramping rate of the generator unit $j \in \mathcal{G}(i, f)$ located at node $i \in \mathcal{N}_f$ owned by firm $f \in \mathcal{F}$ (MW/hr)
α_a	dual variables of transmission constraint in ISO's problem associated with arc a
β_i	dual variables associated with joint sales capacity constraint, one for each node

horizon for our model. Because of the relatively short time-scale, we do not consider the time value of money in our models.

2.3.2. *Revenue and Cost Components* . We consider the inverse demand function for market $i \in \mathcal{N}$ to be

$$\pi_i\left(\sum_{g\in\mathcal{F}} s_i^g, t\right)$$

In particular, we assume that the inverse demand is separable in the sense that the price at market i only depends on the consumption at that market; i.e.,

$$\pi_i\left(\sum_{g\in\mathcal{F}} s_i^g, t\right) = P_{0,i}(t) - \frac{Q_{0,i}(t)}{P_{0,i}(t)} \cdot \left[\sum_{g\in\mathcal{F}} c_i^g\right]$$

where the coefficients $P_{0,i}(t), Q_{0,i}(t)$ vary with time of the day reflecting the pattern of energy-using activities. This results in price rises during high load periods. The *revenue* that firm f is generating at time t is therefore given by

$$\sum_{i\in\mathcal{N}} \pi_i\left(\sum_{g\in\mathcal{F}} s_i^g, t\right) \cdot s_i^f$$

where the summation is over all markets as it is assumed that any generating firm can service any market.

The costs that the generating firm bears are:

1. The *generation cost* (i.e., the cost per unit power generation) for some generation unit $j \in \mathcal{G}(i, f)$ is denoted by

$$V_j^f\left(q_j^f, t\right)$$

and typically has a fixed component and a variable component. Generation cost is usually quadratic of the form

$$V_j^f\left(q_j^f, t\right) = \mu_j^f + \tilde{\mu}_j^f \cdot q_j^f + \frac{1}{2}\hat{\mu}_j^f \cdot \left(q_j^f\right)^2$$

where $\mu_j^f, \tilde{\mu}_j^f, \hat{\mu}_j^f \in \mathbb{R}^1_{++}$ for all $f \in \mathcal{F}, j \in \mathcal{G}(i, f)$.

2. The *ramping cost* is obtained from the fatigue effect of the rotors (which in turn affects the life span of the rotor). However, the ramping cost is negligible if the magnitude of power change is less than some elastic range; that is, there is a range in which the generation rate can be adjusted that causes minimal wear on the rotors and is thus considered cost free. The slope of the cost curve also depends on the ramp-up time. Therefore, in general we may use the function
$$\Phi_j^f\left(r_j^f,t\right)=\frac{1}{2}\gamma_j^f\left[\max\left(0,\left|r_j^f\right|-\xi_j^f\right)\right]^2$$
to represent the ramping cost associated with some generation unit $j\in\mathcal{G}(i,f)$ when the ramping rate is r_j^f, $\xi_j^f\in\mathbb{R}_{++}^1$ is the elastic threshold of the unit and γ_j^f is the cost coefficient which depends on the ramp-up time. In this case, we are using a symmetric cost for ramping up and ramping down, though this is not necessary in general. In general, with asymmetric ramp-up and ramp-down costs we may have
$$\Phi_j^f\left(r_j^f,t\right)=\frac{1}{2}\gamma_j^{f+}\left[\max\left(0,r_j^f-\xi_j^{f+}\right)\right]^2+\frac{1}{2}\gamma_j^{f-}\left[\max\left(0,-r_j^f+\xi_j^{f-}\right)\right]^2$$
where $\gamma_j^{f-},\gamma_j^{f+}$ are the cost coefficients during ramp-up and ramp-down respectively and ξ_j^{f+},ξ_j^{f-} are the elastic thresholds during ramp-up and ramp-down respectively.
3. The *wheeling fee* $w_i(t)$ is paid to the ISO for transmitting 1 MW-hour of power from the hub to market i at time t. The wheeling fee is endogenously determined to enforce market clearing for transmission capacity (i.e., the demand for transmission capacity can be no more than the capacity available).

2.3.3. *Constraints.*

1. Each firm must balance sales and generation at each time $t\in[t_0,t_1]$ as we do not consider storage of electricity
$$\sum_{i\in\mathcal{N}}s_i^f(t)=\sum_{i\in\mathcal{N}_f}\sum_{j\in\mathcal{G}(i,f)}q_j^f(t)$$
2. The sales of power at every market must be nonnegative:
$$s_i^f(t)\geq 0\quad\text{for all } t\in[t_0,t_1]$$
3. The output level of each generating unit is bounded from above and below as
$$0\leq q_j^f(t)\leq CAP_j^f$$
for all $i\in\mathcal{N}_f$, $j\in\mathcal{G}(i,f)$ and $t\in[t_0,t_1]$ where $CAP_j^f\in\mathbb{R}_{++}^1$. The upper bound is imposed by the physical constraints of the generator.
4. Sales may be bounded by local transmission constraints not explicitly represented in the model. This can be represented by the constraint of the form
$$\sum_{f\in\mathcal{F}}s_i^f(t)\leq\sigma_i\quad\text{for all } i\in\mathcal{N},\ t\in[t_0,t_1]$$
where $\sigma_i\in\mathbb{R}_{++}^1$ is the regional sales cap. Note that this constraint makes the equilibrium problem a generalized Nash equilibrium problem, as with this constraint each firms' strategy set (here for sales variable) depends on the rival players' strategy (competitors sales variable).

5. The ramping rate for every generation unit is bounded from above and below,
$$R_j^{f-} \leq r_j^f(t) \leq R_j^{f+}$$
for all $i \in \mathcal{N}_f$, $j \in \mathcal{G}(i,f)$ and $t \in [t_0, t_1]$

2.3.4. *Firms' Extremal Problem.* With the transmission (wheeling) fee w_i and the rival firms' sales
$$s^{-f} \equiv \{s^g : g \in \mathcal{F} \backslash f\}$$
taken as exogenous to the firm $f \in \mathcal{F}$'s optimal control problem and yet endogenous to the overall equilibrium model, firm f computes its nodal sales s^f and generations q^f in order to :

$$\max J\left(s^f, q^f; s^{-f}; t\right) = \int_{t_0}^{t_1} \left\{ \sum_{i \in \mathcal{N}} \pi_i \left(\sum_{g \in \mathcal{F}} s_i^g, t \right) \cdot s_i^f \right. \tag{1}$$

$$\left. - \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} \left[V_j^f\left(q_j^f, t\right) + \Phi_j^f\left(r_j^f, t\right) \right] - \sum_{i \in \mathcal{N}} w_i \cdot \left(s_i^f - \sum_{j \in \mathcal{G}(i,f)} q_j^f \right) \right\} dt$$

subject to

$$\frac{dq_j^f}{dt} = r_j^f \quad \text{for all } i \in \mathcal{N}_f, j \in \mathcal{G}(i,f) \tag{2}$$
$$q_j^f(t_0) = q_{j,0}^f \in \mathbb{R}_+^1 \text{ for all } i \in \mathcal{N}_f, j \in \mathcal{G}(i,f) \tag{3}$$
$$R_j^{f-} \leq r_j^f \leq R_j^{f+} \quad \text{for all } i \in \mathcal{N}_f, j \in \mathcal{G}(i,f) \tag{4}$$
$$0 \leq q_j^f \leq CAP_j^f \quad \text{for all } i \in \mathcal{N}_f, j \in \mathcal{G}(i,f) \tag{5}$$
$$s_i^f \geq 0 \quad \text{for all } i \in \mathcal{N} \tag{6}$$
$$s_i^f + \sum_{g \in \mathcal{F} \backslash i} s_i^g \leq \sigma_i \quad \text{for all } i \in \mathcal{N} \tag{7}$$
$$\sum_{i \in \mathcal{N}} s_i^f = \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} q_j^f \tag{8}$$

Let us review the formulation from an optimal control theory perspective. Constraints of type (8) are called the mixed constraints as they involve both states (q_j^f) and controls (s_i^f). (4) and (6) are the pure control constraints imposing lower and upper bounds on the ramping rates and nonnegativity of sales quantity respectively. (5) are the state-space constraints. The right hand side of the dynamics, (2), are free from states and linear in controls (ramping rates).

3. **The Differential Quasi Variational Inequality (DQVI) formulation.** Let us begin by noting that (1) - (8) form a standard form optimal control problem for firm f, where the controls for the firm are
$$s^f = \left\{ s_i^f : i \in \mathcal{N} \right\}$$
$$r^f = \left\{ r_j^f : i \in \mathcal{N}_f, \ j \in \mathcal{G}(i,f) \right\}$$
and the states are
$$q^f = \left\{ q_j^f : i \in \mathcal{N}_f, \ j \in \mathcal{G}(i,f) \right\}$$

Note that we have state space constraints in the form

$$0 \leq q^f \leq CAP^f$$

where

$$CAP^f = \left\{ CAP_j^f : i \in \mathcal{N}_f,\ j \in \mathcal{G}(i,f) \right\}$$

are known constants. The Hamiltonian associated with this optimal control problem is

$$H_f\left(s^f, r^f; q^f; s^{-f}; \lambda^f, t\right) = \sum_{i \in \mathcal{N}} \pi_i \left(\sum_{g \in \mathcal{F}} s_i^g, t \right) \cdot s_i^f - \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} V_j^f\left(q_j^f, t\right)$$

$$- \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} \Phi_j^f\left(r_j^f, t\right) - \sum_{i \in \mathcal{N}} w_i \cdot \left(s_i^f - \sum_{j \in \mathcal{G}(i,f)} q_j^f \right) + \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} \lambda_j^f \cdot r_j^f \quad (9)$$

where the feasible set of controls for firm f is expressed as

$$\Omega_f = \left\{ \left(s^f, r^f\right) : (4) \text{ - } (8) \text{ hold} \right\}$$

Note that we have not yet explicitly considered the state space constraints and the mixed constraints in the Hamiltonian. This is accomplished by forming the Lagrangian as (see Sethi and Thompson [20])

$$\begin{aligned}
& L_f\left(s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^{f-}, \tau^{f+}, \varsigma^{f-}, \varsigma^{f+}; t\right) \\
= \; & H_f\left(s^f, r^f; q^f; s^{-f}; \lambda^f, t\right) \\
& + \tau^{f-} \left(\sum_{i \in \mathcal{N}} s_i^f - \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} q_j^f \right) \\
& + \tau^{f+} \left(- \sum_{i \in \mathcal{N}} s_i^f + \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} q_j^f \right) \qquad (10) \\
& + \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} \left\{ \varsigma_j^{f-} \cdot q_j^f + \varsigma_j^{f+} \cdot \left(CAP_j^f - q_j^f \right) \right\}
\end{aligned}$$

where $\tau^{f-} \in \mathbb{R}_+^1, \tau^{f+} \in \mathbb{R}_+^1, \varsigma_j^{f-} \in \mathbb{R}_+^1$ and $\varsigma_j^{f+} \in \mathbb{R}_+^1$ are dual variables. τ^{f-} and τ^{f+} satisfy the complementarity slackness conditions

$$\tau^{f-} \geq 0, \quad \tau^{f-} \left(\sum_{i \in \mathcal{N}} s_i^f - \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} q_j^f \right) = 0 \qquad (11)$$

$$\tau^{f+} \geq 0, \quad \tau^{f-} \left(- \sum_{i \in \mathcal{N}} s_i^f + \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} q_j^f \right) = 0 \qquad (12)$$

Also, ς_j^{f-} and ς_j^{f+} satisfy the conditions

$$\varsigma^{f-} \geq 0, \sum_{i\in\mathcal{N}_f}\sum_{j\in\mathcal{G}(i,f)} \varsigma_j^{f-}\cdot q_j^f = 0, \frac{d\varsigma^{f-}}{dt} \leq 0 \tag{13}$$

$$\varsigma^{f+} \geq 0, \sum_{i\in\mathcal{N}_f}\sum_{j\in\mathcal{G}(i,f)} \varsigma_j^{f+}\cdot \left(CAP_j^f - q_j^f\right) = 0, \frac{d\varsigma^{f+}}{dt} \leq 0 \tag{14}$$

where

$$\begin{aligned}\varsigma^{f-} &= \left\{\varsigma_j^{f-} : i\in\mathcal{N}_f,\ j\in\mathcal{G}(i,f)\right\}\\ \varsigma^{f+} &= \left\{\varsigma_j^{f+} : i\in\mathcal{N}_f,\ j\in\mathcal{G}(i,f)\right\}\end{aligned}$$

Note that these conditions on the multipliers ς^{f-} and ς^{f+} arise from the state space constraints, whereas (11) - (12) arise from the mixed constraints.

The adjoint variables λ^f follow the dynamics

$$\begin{aligned}\frac{d\lambda_j^{*f}}{dt} &= -\frac{\partial L_f^*}{\partial q_j^f},\ \forall i\in\mathcal{N}_f,\ j\in\mathcal{G}(i,f)\\ &= \frac{\partial\Phi_j^f\left(r_j^{f*},t\right)}{\partial q_j^f} + \tau^{f-} - \tau^{f+} - \varsigma_j^{f-} + \varsigma_j^{f+},\ \forall i\in\mathcal{N}_f,\ j\in\mathcal{G}(i,f) \end{aligned}\tag{15}$$

and the transversality condition yields

$$\lambda_j^{*f}(t_1) = 0 \tag{16}$$

Therefore, for given set of controls r^f, the state variables $\left(q^f\right)$ and adjoint variables $\left(\lambda^f\right)$ can be determined in a sequential fashion; note that this is not an approximation. This observation will facilitate our computational efforts.

Also note that constraints (7) are joint constraints involving s^f and s^{-f}. We handle these constraints in the following way as outlined by Pang [17], although we are applying this to a dynamic game.

From the maximum principle (see Bryson and Ho [2]) we know that the necessary condition (which we will later establish to be a sufficient condition as well) for the quadruplet $\left\{s^{f*}(t), r^{f*}(t); q^{f*}(t); \lambda^{f*}(t)\right\}$ (as well as the dual multipliers $\tau^{f-}, \tau^{f+}, \varsigma^{f-}, \varsigma^{f+}$) being an optimal solution to the optimal control problem (1) - (8) is that the nonlinear program

$$\max L_f\left(s^f, r^f; q^f; s^{-f}; \lambda^f, t\right)$$

$$\text{subject to } \left\{s^f(t), r^f(t)\right\} \in \tilde{\Omega}_f$$

be solved for each instant $t\in[t_0,t_1]$ where $\tilde{\Omega}_f$ is a set formed by pure control constraints

$$\tilde{\Omega}_f = \left\{\left(s^f, r^f\right) : (4)\text{-}(8) \text{ hold}\right\}$$

This is a stationary problem. Further, we write this problem as

$$\max L_f\left(s^f, r^f; q^f; s^{-f}; \lambda^f, t\right) \tag{17}$$

subject to

$$\begin{aligned} s^f + \sum_{g \in \mathcal{F} \setminus i} s^g &\leq \sigma & (18)\\ R^{f-} &\leq r^f \leq R^{f+} & (19)\\ s^f &\geq 0 & (20) \end{aligned}$$

Note that, even though we only have control constraints in (18), (19) and (20), constraint (18) is a joint constraint which is problematic. Under a suitable constraint qualification such as Abadie's (as the feasible set is convex), we obtain the following KKT conditions which are necessary for s^f being optimal for firm f's stationary problem (17) - (18).

$$\begin{aligned} 0 &= -\nabla_{s^f} \tilde{L}_f \left(s^f, r^f; q^f; s^{-f}; \lambda^f, t \right) - \gamma^f & (21)\\ 0 &\leq \beta \perp \left(s^f + \sum_{g \in \mathcal{F} \setminus i} s^g - \sigma \right) \leq 0 & (22)\\ 0 &\leq \gamma^f \perp \left(-s^f \right) \leq 0 & (23) \end{aligned}$$

where $\tilde{L}_f \left(s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \varsigma^{f-}, \varsigma^{f+}, \beta; t \right)$ is agent f's 'semi-Lagrangian function' defined as

$$\begin{aligned} & \tilde{L}_f \left(s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \varsigma^{f-}, \varsigma^{f+}, \beta; t \right) & (24)\\ = \ & L_f \left(s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \varsigma^{f-}, \varsigma^{f+}; t \right) \\ & + \beta^T \cdot \left(s^f + \sum_{g \in \mathcal{F} \setminus i} s^g - \sigma \right) \end{aligned}$$

Note that, as discussed in Harker [9], we use the restricted multiplier formulation which is applicable to all generalized Nash games where the players have the same joint constraint. In this case we use dual multiplier β_i associated with the regional sales cap constraint. This dual multiplier β_i depends only on the region (market) i and is common to all firms.

Therefore, concatenating $|\mathcal{F}|$ number of KKT systems, we obtain a characterization of a generalized Cournot-Nash equilibrium as a partitioned DVI. In order to formulate the latter DVI, let us define the function $\mathbf{\Theta} \left(s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \varsigma^{f-}, \varsigma^{f+}; t \right)$ as

$$\begin{aligned} \mathbf{\Theta} \left(s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \varsigma^{f-}, \varsigma^{f+}; t \right) &= \begin{pmatrix} \left(\nabla_{s^f} L_f (\cdot) + \beta \right)_{f=1}^{|\mathcal{F}|} \\ \left(s^f + \sum_{g \in \mathcal{F} \setminus i} s^g - \sigma \right) \end{pmatrix} \\ z &= \begin{pmatrix} \left(s^f \right)_{f=1}^{|\mathcal{F}|} \\ \beta \end{pmatrix} \end{aligned}$$

Further, considering the complementarity conditions (11)-(14), we can define $\tilde{\mathbf{\Theta}}(\cdot)$ as

$$\tilde{\mathbf{\Theta}}(\cdot) = \begin{pmatrix} \left(\nabla_{s^f} L_f\left(s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^f, \varsigma^{f-}, \varsigma^{f+}; t\right) + \beta\right)_{f=1}^{|\mathcal{F}|} \\ \left(s^f + \sum_{g \in \mathcal{F} \setminus i} s^g - \sigma\right) \\ \left(-\sum_{i \in \mathcal{N}} s_i^f + \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} q_j^f\right)_{f=1}^{|\mathcal{F}|} \\ \left(\sum_{i \in \mathcal{N}} s_i^f - \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} q_j^f\right)_{f=1}^{|\mathcal{F}|} \\ -\left(q^f\right)_{f=1}^{|\mathcal{F}|} \\ \left(-CAP^f + q^f\right)_{f=1}^{|\mathcal{F}|} \end{pmatrix} \tag{25}$$

$$\tilde{z} = \begin{pmatrix} \left(s^f\right)_{f=1}^{|\mathcal{F}|} \\ \beta \\ \left(\tau^{f-}\right)_{f=1}^{|\mathcal{F}|} \\ \left(\tau^{f+}\right)_{f=1}^{|\mathcal{F}|} \\ \left(\varsigma^{f-}\right)_{f=1}^{|\mathcal{F}|} \\ \left(\varsigma^{f+}\right)_{f=1}^{|\mathcal{F}|} \end{pmatrix} \tag{26}$$

The solution of the following partitioned DVI described below is also the generalized Cournot-Nash equilibrium of the game described above, taking the wheeling fee w as exogenous :

$$\text{find } \left(\tilde{z}^*, r^*\right) \in \Lambda \text{ such that}$$

$$\int_{t_0}^{t_1} \left\{ \begin{array}{c} \left[\tilde{\mathbf{\Theta}}\left(\tilde{z}^*, r^{f*}; t\right)\right]^T \left(\tilde{z} - \tilde{z}^*\right) dt \\ + \sum_{f \in \mathcal{F}} \int_{t_0}^{t_1} \left[\nabla_{r^f} L_f\left(\tilde{z}^*, r^{f*}; t\right)\right]^T \left(r^f - r^{f*}\right) dt \end{array} \right\} \leq 0 \tag{27}$$

$$\text{for all } \begin{pmatrix} r \\ \tilde{z} \end{pmatrix} \in \begin{pmatrix} \Lambda \\ \kappa \end{pmatrix}$$

where

$$\begin{aligned} \Lambda_f &= \left\{r^f : \text{constraints (4) hold}\right\} \\ \kappa &= \left\{\tilde{z} : \tilde{z} \geq 0\right\} \end{aligned}$$

and

$$\Lambda = \prod_{f \in \mathcal{F}} \Lambda_f$$

3.1. **The ISO and transmission fees.** It was mentioned earlier that the transmission (wheeling) fees w are set by the ISO in order to efficiently clear the market for transmission capacity. Specifically, taking w as exogenous to its problem, at every instant $t \in [t_0, t_1]$ the ISO seeks to solve the following linear program to determine the transmission flows y in order to

$$\max J_2(t) = \sum_{i \in \mathcal{N}} y_i(t) \cdot w_i(t) \tag{28}$$

subject to

$$\sum_{i \in \mathcal{N}} PTDF_{ia} \cdot y_i(t) \leq T_a \text{ for all } a \in \mathcal{A} \quad (\alpha_a) \tag{29}$$

where we write dual variables in the parentheses next to the corresponding constraints. Recall that $\mathcal{A}$ is the arc set of the electric power network, T_a is the transmission capacity on arc $a \in \mathcal{A}$ and $PTDF_{ia}$ are the power transmission distribution factors that describe how much MW flow occurs through arc a as a result of a unit MW injection at an arbitrary hub node and a unit withdrawal at node i. In a linearized DC power flow model, which is the basis of the above model, the PTDF factors are assumed to be constant and are unaffected by the load of the transmission line. Therefore the principle of superposition applies. The decision variables $y_i(t)$ denote transfers of power in MW by the ISO from a hub node to the node (market) $i \in \mathcal{N}$ at time $t \in [t_0, t_1]$. In this particular formulation we ignore transmission losses, however, our model is general enough and does not prohibit us from considering non-linear losses. In the case of losses, either the ISO or the firms involved in the transaction should account for the losses and a book-keeping effort is required. Furthermore, Kirchhoff's current and voltage laws would need to be explicitly stated because superposition no longer applies when losses are non-linear and the PTDFs are no longer constant.

3.2. **The market clearing conditions.** To clear the market, the transmission flows y_i must balance the net sales at each node (market), therefore

$$y_i(t) = \sum_{f \in \mathcal{F}} \left(s_i^f(t) - \sum_{j \in \mathcal{G}(i,f)} q_j^f(t) \right) \quad \text{for all } i \in \mathcal{N}$$

Therefore, re-writing (28) - (29) we have

$$\max J_2(t) = \sum_{i \in \mathcal{N}} \sum_{f \in \mathcal{F}} \left(s_i^f(t) - \sum_{j \in \mathcal{G}(i,f)} q_j^f(t) \right) \cdot w_i(t) \tag{30}$$

subject to

$$\sum_{i \in \mathcal{N}} PTDF_{ia} \cdot \sum_{f \in \mathcal{F}} \left(s_i^f(t) - \sum_{j \in \mathcal{G}(i,f)} q_j^f(t) \right) \leq T_a \text{ for all } a \in \mathcal{A} \quad (\alpha_a) \tag{31}$$

Therefore, the optimality condition of the linear program (30) - (31) can be written down as

$$\begin{aligned} w_i &= \sum_{a \in \mathcal{A}} PTDF_{ik} \cdot \alpha_a(t) \quad \forall i \in \mathcal{N} \\ 0 &\leq \alpha_a(t) \perp T_a - \sum_{i \in \mathcal{N}} PTDF_{ia} \cdot \sum_{f \in \mathcal{F}} \left(s_i^f(t) - \sum_{j \in \mathcal{G}(i,f)} q_j^f(t) \right) \geq 0, \ \forall a \in \mathcal{A} \end{aligned}$$

To articulate the complete DQVI formulation, let us define the following vector

$$v(s, q; t) = \left(T_a - \sum_{i \in \mathcal{N}} PTDF_{ia} \cdot \sum_{f \in \mathcal{F}} \left(s_i^f(t) - \sum_{j \in \mathcal{G}(i,f)} q_j^f(t) \right) \right)_{a=1}^{|\mathcal{A}|}$$

3.3. **The complete restricted DVI formulation.** Putting together the generating firms' optimality conditions, the ISO's problem and the market clearing condition, we obtain the complete formulation of the market equilibrium problem as the following DQVI :

$$\text{find } (z^*, r^*, \alpha^*) \in \tilde{\Lambda} \text{ such that}$$

$$\int_{t_0}^{t_1} \left\{ \left[\tilde{\mathbf{\Theta}} \left(\tilde{z}^*, r^{f*}; t \right) \right]^T (\tilde{z} - \tilde{z}^*)\, dt + \right. \tag{32}$$

$$\left. \sum_{f \in \mathcal{F}} \int_{t_0}^{t_1} \left[\nabla_{r^f} L_f \left(\tilde{z}^*, r^{f*}; t \right) \right]^T \left(r^f - r^{f*} \right) \right\} dt$$

$$- \int_{t_0}^{t_1} \left\{ \upsilon \left(s^*, q^*; t \right) \cdot (\alpha - \alpha^*) \right\} dt \leq 0$$

$$\text{for all } (z, r, \alpha) \in \tilde{\Lambda}$$

where

$$\tilde{H}_f \left(s^f, r^f; q^f; s^{-f}; \lambda^f, \alpha, t \right) = \sum_{i \in \mathcal{N}} \pi_i \left(\sum_{g \in \mathcal{F}} s_i^g, t \right) \cdot s_i^f$$
$$- \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} \left[V_j^f \left(q_j^f, t \right) + \Phi_j^f \left(r_j^f, t \right) \right]$$
$$- \sum_{i \in \mathcal{N}} \left(\sum_{a \in \mathcal{A}} PTDF_{ik} \cdot \alpha_a (t) \right) \cdot \left(s_i^f - \sum_{j \in \mathcal{G}(i,f)} q_j^f \right) + \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} \lambda_j^f \cdot r_j^f$$

and $L_f \left(s^f, r^f; q^f; s^{-f}; \lambda^f, \tau^{f-}, \tau^{f+}, \varsigma^{f-}, \varsigma^{f+}; t \right)$ and $\tilde{\mathbf{\Theta}} \left(s^{f*}, r^{f*}; q^{f*}; s^{-f*}; \lambda^{f*}, t \right)$ are defined in the same way as in (10) and (25) respectively, only replacing the Hamiltonian $H_f(\cdot)$ by the augmented Hamiltonian $\tilde{H}_f(\cdot)$. Further, $\tilde{\Lambda}$ is defined as

$$\tilde{\Lambda}_f = \left\{ \left(\tilde{z}, r^f, \alpha \right) : \begin{array}{c} \tilde{z} \geq 0 \\ R^{f-} \leq r^f \leq R^{f+} \\ \alpha \geq 0 \\ \dot{\varsigma}^{f-} \leq 0 \\ \dot{\varsigma}^{f+} \leq 0 \end{array} \right\}$$
$$\tilde{\Lambda} = \prod_{f \in \mathcal{F}} \tilde{\Lambda}_f$$

Note that the last two constraints can be rewritten as

$$\frac{d\varsigma^{f-}}{dt} = u^{f-} \tag{33}$$
$$\frac{d\varsigma^{f+}}{dt} = u^{f+} \tag{34}$$
$$u^{f-} \leq 0$$
$$u^{f+} \leq 0$$

where u^{f-} and u^{f+} are dummy control variables. We treat these dynamics explicitly in our numerical algorithm (implicit fixed point algorithm) as discussed in Section 4.

3.4. **Equivalence between the GNEP and the DVI (32).** Before we establish the equivalence between the GNEP and the restricted DVI formulation provided in (32), we need to establish that the necessary conditions for the optimal control problem (1) - (8) are sufficient conditions as well. The lemma stated below establishes the sufficiency condition.

Lemma 3.1. *(Sufficiency conditions for the optimal control problem) For a given* s^{-f} *let* $\left(s^{f*}, r^{f*}; q^{f*}; s^{-f}; \lambda^f, \tau^{f-}, \tau^{f+}, \varsigma^{f-}, \varsigma^{f+}\right)$ *satisfy the necessary conditions (11)-(20), then* $\left(s^{f*}, r^{f*}; q^{f*}\right)$ *is optimal*

Proof. We begin with defining an augmented adjoint variable

$$\begin{aligned}\tilde{\lambda}_j^f(t) &= \lambda_j^f(t) + \varsigma_j^{f-}\frac{\partial q_j^f}{\partial q_j^f} + \varsigma_j^{f+}\frac{\partial\left(CAP_j^f - q_j^f\right)}{\partial q_j^f} \\ &= \lambda_j^f(t) + \varsigma_j^{f-} - \varsigma_j^{f+}\end{aligned}$$

for all $i \in \mathcal{N}_f,\ j \in \mathcal{G}(i,f)$ and $t \in [t_0, t_1]$. We also observe that for a given s^{-f}, the instantaneous revenue is concave in the generating firm's own sales s^f, the unit generation cost is convex in q^f, the unit ramping cost is also convex in ramping rate r^f and the transshipment cost is linear in s^f and q^f. Also, the right hand side of the dynamics (2) of firm f is linear in the firm's own ramping rate r^f and the expression for $\tilde{\lambda}_j^f$ does not involve state variables. Using the fact that the negative of a convex function is a concave function, we observe that the modified Hamiltonian $\tilde{H}_f\left(s^f, r^f; q^f; s^{-f}; \tilde{\lambda}^f, t\right)$

$$\tilde{H}_f(\cdot) = \sum_{i\in\mathcal{N}}\pi_i\left(\sum_{g\in\mathcal{F}} s_i^g, t\right)\cdot s_i^f - \sum_{i\in\mathcal{N}_f}\sum_{j\in\mathcal{G}(i,f)}\left[V_j^f\left(q_j^f, t\right) + \Phi_j^f\left(r_j^f, t\right)\right]$$

$$-\sum_{i\in\mathcal{N}}\left(\sum_{a\in\mathcal{A}} PTDF_{ik}\cdot\alpha_a(t)\right)\cdot\left(s_i^f - \sum_{j\in\mathcal{G}(i,f)} q_j^f\right) + \sum_{i\in\mathcal{N}_f}\sum_{j\in\mathcal{G}(i,f)}\tilde{\lambda}_j^f\cdot r_j^f$$

is concave in $\left(s^f, r^f; q^f\right)$ at each $t \in [t_0, t_1]$. Also observe that the state-space constraints (5) are linear constraints as are the mixed constraints (8). Further, there is no terminal state constraint. Hence we can apply the Theorem 4.1 of Sethi and Thompson [20] which ensures optimality of $\left(s^{f*}, r^{f*}; q^{f*}\right)$. Hence the proof. $\square$

We now state and establish the equivalence between the GNEP and the DVI (32).

Theorem 3.2. *(Equivalence between the GNEP and DVI) Under a suitable constraint qualification (e.g., for every* f *and* $t \in [t_0, t_1]$*, there exists* $\tilde{s}^f$*,* $\tilde{r}^f$ *and* $\tilde{q}^f$ *such that* $\tilde{q}^f > 0$*,* $\tilde{q}^f < CAP^f$*,* $\tilde{s}^f + \sum_{g\in\mathcal{F}\setminus i} s^{g*} < \sigma$*,* $\tilde{s}^f > 0$*,* $\sum_{i\in\mathcal{N}} s_i^f > \sum_{i\in\mathcal{N}_f}\sum_{j\in\mathcal{G}(i,f)} q_j^f$ *and* $\sum_{i\in\mathcal{N}} s_i^f < \sum_{i\in\mathcal{N}_f}\sum_{j\in\mathcal{G}(i,f)} q_j^f$*), the tuple* (s^*, q^*) *is a GNE if and only if there exists* $\beta^*, \tau^{-*}, \tau^{+*}, \varsigma^{-*}, \varsigma^{+*}, \alpha^*$ *such that a solution of the DVI (32) is* $(\beta^*, \tau^{-*}, \tau^{+*}, \varsigma^{-*}, \varsigma^{+*}, \alpha^*, s^*, q^*)$*.*

Proof. We begin by noting that (32) is equivalent to the following optimal control problem

$$\max J_3\left(z, r^f, r^{-f}, \alpha, t\right) = \int_{t_0}^{t_1}\left\{\begin{array}{l}\left[\tilde{\boldsymbol{\Theta}}\left(\tilde{z}^*, r^{f*}, \alpha^*; t\right)\right]^T\cdot\tilde{z} \\ +\left[\nabla_{r^f} L_f\left(\tilde{z}^*, r^{f*}; t\right)\right]^T\cdot r^f \\ -\upsilon\left(s^*, q^*; t\right)\cdot\alpha\end{array}\right\}dt \qquad (35)$$

$$\text{s.t.} \qquad (z.r, \alpha) \in \tilde{\Lambda} \qquad (36)$$

where it is essential to recognize that $J_3\left(z, r^f, r^{-f}, \alpha, t\right)$ is a 'linear' functional that assumes knowledge of the solution to our oligopolistic game; as such, $J_3\left(z, r^f, r^{-f}, \alpha, t\right)$ is a mathematical construct for use in analysis and has no meaning as a computational device. The augmented Hamiltonian for this artificial optimal control problem is

$$\begin{aligned} H_0 &= \left[\tilde{\mathbf{\Theta}}\left(\tilde{z}^*, r^{f*}, \alpha^*; t\right)\right]^T \cdot \tilde{z} + \left[\nabla_{r^f} L_f\left(\tilde{z}^*, r^{f*}; t\right)\right]^T \cdot r^f \\ &\quad -\upsilon\left(s^*, q^*; t\right) \cdot \alpha + \sum_{f \in \mathcal{F}} \left[\mu^{f-}\right]^T u^{f-} + \sum_{f \in \mathcal{F}} \left[\mu^{f+}\right]^T u^{f+} \end{aligned}$$

where μ^{f-} and μ^{f+} are the auxiliary adjoint variables associated with the dynamics (33) and (34) respectively. The associated maximum principal requires

$$\max H_0$$

subject to

$$\begin{aligned} R^- &\leq r \leq R^+ \\ \tilde{z} &\geq 0 \\ \alpha &\geq 0 \end{aligned}$$

for all $t \in [t_0, t_1]$. The corresponding necessary and sufficient (as the Hamiltonian is linear in the controls) conditions for this optimal control problem are identical to (37) through (39):

$$\nabla_{\tilde{z}} H_0^* = \tilde{\mathbf{\Theta}}\left(\tilde{z}^*, r^{f*}, \alpha^*; t\right) \tag{37}$$

$$\nabla_{r^f} H_0^* = \nabla_{r^f} L_f\left(\tilde{z}^*, r^{f*}; t\right) \tag{38}$$

$$\nabla_{\alpha} H_0^* = -\upsilon\left(s^*, q^*; t\right) \tag{39}$$

where

$$\begin{aligned} H_0^* &= \left[\tilde{\mathbf{\Theta}}\left(\tilde{z}^*, r^{f*}, \alpha^*; t\right)\right]^T \cdot \tilde{z}^* + \sum_{f \in \mathcal{F}} \left[\nabla_{r^f} L_f\left(\tilde{z}^*, r^{f*}; t\right)\right]^T \cdot r^{f*} \\ &\quad -\upsilon\left(s^*, q^*; t\right) \cdot \alpha^* + \sum_{f \in \mathcal{F}} \left[\mu^{f-}\right]^T u^{f-*} + \sum_{f \in \mathcal{F}} \left[\mu^{f+}\right]^T u^{f+*} \end{aligned}$$

since these are identical to the necessary conditions for the firm f's optimal control problem. Further, from Lemma 3.1 we know these are the sufficient conditions for firm f's optimal control problem, hence the desired result (32) is immediate. □

3.5. **Existence.** We also establish the existence of a solution of the GNEP. Since we have already established in Theorem 3.2 the equivalence between the GNEP and the DVI (32), it is sufficient to show that the DVI (32) has a solution. We state and prove the following result.

Theorem 3.3. *(Existence of a solution of the GNEP) Under a suitable constraint qualification (e.g., for every f and $t \in [t_0, t_1]$, there exists $\tilde{s}^f$, $\tilde{r}^f$ and $\tilde{q}^f$ such that $\tilde{q}^f > 0$, $\tilde{q}^f < CAP^f$, $\tilde{s}^f + \sum_{g \in \mathcal{F} \setminus i} s^{g*} < \sigma$, $\tilde{s}^f > 0$, $\sum_{i \in \mathcal{N}} s_i^f > \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} q_j^f$ and $\sum_{i \in \mathcal{N}} s_i^f < \sum_{i \in \mathcal{N}_f} \sum_{j \in \mathcal{G}(i,f)} q_j^f$), the GNEP has a solution*

Proof. The formal proof of this existence result is in Bowder [1], and depends on the conversion of the DVI to a fixed point problem involving the minimum norm projection and application of Browder's existence theorem. We know $q(r, t)$ is well defined and continuous. So the principal functions of the DVI, i.e., $\tilde{\mathbf{\Theta}}(\cdot)$, $\nabla_{r^f} L_f(\cdot)$

and $\upsilon(\cdot)$ are continuous in the controls s and r. Also since the feasible control set for each player is convex and compact, their Cartesian products is also convex and compact. Consequently, by Theorem 2 of Bowder [1], DVI (32) has a solution. Further, under a suitable constraint qualification, we can invoke the Theorem 3.2 which ascertains that the GNEP also has a solution. Hence the proof. □

4. **Algorithms for the DVI.** We employ a version of an implicit fixed point algorithm to compute the equilibria of the model. There is a fixed point form of $DVI(F, f, U, x^0, D)$. In particular we state and prove the following result:

Theorem 4.1. *(fixed point formulation of the DVI (32)) The DVI (32) is equivalent to the following fixed point problem:*

$$\begin{pmatrix} \tilde{z} \\ r \\ \alpha \end{pmatrix} = P_{\tilde{\Lambda}}\left[\begin{pmatrix} \tilde{z} \\ r \\ \alpha \end{pmatrix} - \alpha_{fp}\begin{pmatrix} \tilde{\mathbf{\Theta}}\left(\tilde{z}, r^f, \alpha; t\right) \\ \nabla_{r^f} L_f\left(\tilde{z}, r^f; t\right) \\ \upsilon\left(s; t\right) \end{pmatrix}\right]$$

where $P_{\tilde{\Lambda}}[.]$ is the minimum norm projection onto $\tilde{\Lambda}$ and $\alpha_{fp} \in \Re^1_{++}$.

Proof. We observe that the right hand side of the dynamics (2) for each generator's optimal control problem is linear in controls (r^f) and is thus convex. Further, the principal functions of the DVI, namely $\tilde{\mathbf{\Theta}}\left(\tilde{z}, r^f, \alpha; t\right)$, $\nabla_{r^f} L_f\left(\tilde{z}, r^f; t\right)$ and $\upsilon(s, q; t)$ are continuous with respect to both the controls (s^f, r^f) and states (q^f); hence, all the regularity conditions of Definition 2 of Friesz and Mookherjee [6] hold. Using Theorem 3 of Friesz and Mookherjee [6] we obtain the desired result. □

Naturally there is an associated fixed point algorithm based on the iterative scheme

$$\begin{pmatrix} \tilde{z}^{k+1} \\ r^{k+1} \\ \alpha^{k+1} \end{pmatrix} = P_{\tilde{\Lambda}}\left[\begin{pmatrix} \tilde{z}^k \\ r^k \\ \alpha^k \end{pmatrix} - \alpha_{fp}\begin{pmatrix} \tilde{\mathbf{\Theta}}\left(\tilde{z}^k, r^{f,k}, \alpha^k; t\right) \\ \nabla_{r^f} L_f\left(\tilde{z}^k, r^{f,k}; t\right) \\ \upsilon\left(s^k; t\right) \end{pmatrix}\right]$$

where k in the superscript denotes the counter for the fixed point iteration. The detailed structure of the fixed point algorithm is as follows:

Fixed Point Algorithm

Step 0. Initialization: identify an initial feasible solution $\begin{pmatrix} \tilde{z}^0 \\ r^0 \\ \alpha^0 \end{pmatrix} \in \tilde{\Lambda}$ and set $k = 0$.

Step 1. Solve optimal control subproblem:

$$\min_{(\tilde{z}, r, \alpha)} J^k\left(\tilde{z}, r, \alpha\right) = \frac{1}{2}\int_{t_0}^{t_1}\begin{bmatrix} \left(\tilde{z}^k - \alpha_{fp}\tilde{\mathbf{\Theta}}\left(\tilde{z}^k, r^{f,k}, \alpha^k; t\right) - \tilde{z}\right)^2 \\ + \left(r^k - \alpha_{fp}\nabla_{r^f} L_f\left(\tilde{z}^k, r^{f,k}; t\right) - r\right)^2 \\ + \left(\alpha^k - \alpha_{fp}\upsilon\left(s^k; t\right) - \alpha\right)^2 \end{bmatrix} dt \quad (40)$$

$$\text{subject to } \frac{d\varsigma^-}{dt} = u^-, \ \frac{d\varsigma^+}{dt} = u^+ \quad (41)$$

$$(42)$$

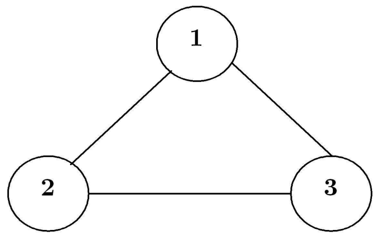

FIGURE 3. 3 node, 3 arc power network with 3 firms each having 6 generating units

$$\begin{aligned} u^- &\leq 0,\ u^+ \leq 0 && (43)\\ R^- &\leq r \leq R^+ && (44)\\ \tilde{z} &\geq 0 && (45)\\ \alpha &\geq 0 && (46)\\ \varsigma^-(t_0) &= \varsigma^{0-}, \varsigma^+(t_0) = \varsigma^{0+} && (47) \end{aligned}$$

Call the solution $\begin{pmatrix} \tilde{z}^{k+1} \\ r^{k+1} \\ \alpha^{k+1} \end{pmatrix}$

Step 2. Stopping test: if

$$\left\| \begin{pmatrix} \tilde{z}^{k+1} \\ r^{k+1} \\ \alpha^{k+1} \end{pmatrix} - \begin{pmatrix} \tilde{z}^{k} \\ r^{k} \\ \alpha^{k} \end{pmatrix} \right\| \leq \varepsilon$$

where $\varepsilon \in \Re^1_{++}$ is a preset tolerance, stop and declare $\begin{pmatrix} \tilde{z}^{*} \\ r^{*} \\ \alpha^{*} \end{pmatrix} \approx \begin{pmatrix} \tilde{z}^{k+1} \\ r^{k+1} \\ \alpha^{k+1} \end{pmatrix}$. Otherwise set $k = k+1$ and and go to Step 1.

The convergence of this algorithm is guaranteed if $\tilde{\mathbf{\Theta}}\left(\tilde{z}, r^f, \alpha; t\right), \left[\nabla_{r^f} L_f\left(\tilde{z}, r^f; t\right)\right]$ and $\upsilon(s;t)$ are strongly monotonic for all $\begin{pmatrix} \tilde{z} \\ r \\ \alpha \end{pmatrix} \in \tilde{\Lambda}$, see Friesz and Mookherjee [8] for a detailed discussion.

5. Numerical example.

5.1. Description of the network and choice of parameters.

Let us consider a 3 arc, 3 node electric power network where a regional market is located at each node and there are 3 firms engaged in oligopolistic competition. The power network is illustrated in Figure 3.

We assume that each firm has 2 generation units at each of the nodes with different capacities and ramping rates. Therefore, each firm has a total of 6 generation units which are geographically separated. We consider inverse demand parameters (namely price and quantity intercept) to be time varying;

Market, i	1	2	3
$P_{0,i}$	40	35	32
$Q_{0,i}$	5000	4000	6200

and $P_{0,i}(t)$ and $Q_{0,i}(t)$ are estimated from the load profiles of Figure 1 for all $t \in [t_0, t_1]$. PDF values associated with the network are as follows

Arc	$(1,2)$	$(1,3)$	$(2,3)$
Node 1	0.33	0.67	0.33
Node 2	-0.33	0.33	0.67
Node 3	0	0	0

Generation capacities, CAP_j^f (in MW) of different generation units are shown below

Firm 1	Unit 1	Unit 2
Node 1	1000	500
Node 2	750	500
Node 3	800	400

Firm 2	Unit 1	Unit 2
Node 1	750	500
Node 2	500	600
Node 3	400	500

Firm 3	Unit 1	Unit 2
Node 1	600	500
Node 2	1000	400
Node 3	1200	400

It is evident from the above tables that each firm has a mix of high and low capacity generators. The ramping rates, r_j^f, are bounded from above and below. We assume these bounds are symmetric, i.e., $R_j^{f+} = -R_j^{f-}$ for all $f \in \mathcal{F}$, $i \in \mathcal{N}_f$, and $j \in \mathcal{G}(i,f)$. The upper bounds on ramping rates for the generation units are shown below.

Firm 1	Unit 1	Unit 2
Node 1	58	336
Node 2	84	340
Node 3	70	400

Firm 2	Unit 1	Unit 2
Node 1	84	330
Node 2	336	290
Node 3	400	340

Firm 3	Unit 1	Unit 2
Node 1	290	340
Node 2	58	380
Node 3	35	365

If we compare ramping rate bounds and generation capacities of the units, it will be evident that units having higher capacity typically have slower ramping capability on percentage basis and vice versa. Also, ramping costs (\$/MW-hour) associated with the faster ramping machines are higher compared to their slower counterparts. We tabulate unit ramping cost γ_j^f parameters below.

Firm 1	Unit 1	Unit 2
Node 1	2.14	6.82
Node 2	4.50	6.75
Node 3	5.50	8.70

Firm 2	Unit 1	Unit 2
Node 1	4.50	6.72
Node 2	6.75	5.93
Node 3	8.74	6.80

Firm 3	Unit 1	Unit 2
Node 1	6.00	6.90
Node 2	2.20	8.60
Node 3	1.54	8.65

Elastic limits for the generators, ξ_j^f, are usually not very dependent on the capacities of the generators, which is also evident from below where we list values of ξ_j^f for the generators (in MW).

Firm 1	Unit 1	Unit 2
Node 1	65	57
Node 2	60	54
Node 3	62	52

Firm 2	Unit 1	Unit 2
Node 1	60	55
Node 2	61	59
Node 3	51	55

Firm 3	Unit 1	Unit 2
Node 1	55	53
Node 2	65	51
Node 3	67	52

The regional sales capacities in each of the 3 markets are assumed to be

Market, i	1	2	3
Market CAP, σ_i (MW)	3000	3200	2900

Coefficients associated with the linear component of generation costs of the units, μ_j^f (\$/MW), are assumed to be

Firm 1	Unit 1	Unit 2
Node 1	15	15
Node 2	14.5	15
Node 3	14.7	15.2

Firm 2	Unit 1	Unit 2
Node 1	15.2	14.7
Node 2	15.1	14.9
Node 3	15	15.1

Firm 3	Unit 1	Unit 2
Node 1	15	15
Node 2	14.8	14.8
Node 3	15.3	15

We typically assume all coefficients associated with the quadratic component of generation costs of units belonging to a firm to be the same with

$$\begin{aligned}
\hat{\mu}_j^1 &= 0.08 \text{ for all } i \in \mathcal{N}_1, j \in \mathcal{G}(i,1) \\
\hat{\mu}_j^2 &= 0.07 \text{ for all } i \in \mathcal{N}_2, j \in \mathcal{G}(i,2) \\
\hat{\mu}_j^3 &= 0.075 \text{ for all } i \in \mathcal{N}_3, j \in \mathcal{G}(i,3)
\end{aligned}$$

Transmission capacities of the arcs are assumed to be the following

Arc, a	1	2	3
Transmission Capacity, T_a (MW)	130	150	160

Our planning horizon in this example is 24 hours with $t_0 = 0$ and $t_1 = 24$. The initial generation rates at $t_0 = 0$ are

$$\begin{aligned}
q_{j,0}^1 &= 150 \text{ for all } i \in \mathcal{N}_1, j \in \mathcal{G}(i,1) \\
q_{j,0}^2 &= 175 \text{ for all } i \in \mathcal{N}_2, j \in \mathcal{G}(i,2) \\
q_{j,0}^3 &= 160 \text{ for all } i \in \mathcal{N}_3, j \in \mathcal{G}(i,3)
\end{aligned}$$

which implies that at the beginning all the generators for a firm are operating at the same level. This choice is intentional, as we want to study the impact of ramping rates on the generators.

5.2. **Sales, ramping rates, generation rates and market prices.**

5.2.1. *Performance of the Algorithm.* We forgo the detailed symbolic statement of this example and, instead, provide numerical results in graphical form for the solution which was obtained after 399 fixed point iterations. We choose the convergence parameter $\alpha_{fp} = \frac{1}{k}$ where k is the iteration counter, and pre-set tolerance $\epsilon = 0.5$ which are the parameters for the fixed point algorithm (see Mookherjee [14]). In Figure 4 the relative change from one iteration to the next, expressed as

$$\Delta_k = \left\| \begin{pmatrix} \tilde{z}^{k+1} \\ r^{k+1} \\ \alpha^{k+1} \end{pmatrix} - \begin{pmatrix} \tilde{z}^{k} \\ r^{k} \\ \alpha^{k} \end{pmatrix} \right\|$$

is plotted against the iteration counter k. It is worth noting that for this particular example even though $\Delta_1 = 5816.9$, in the next several iterations Δ_k decreases very rapidly. The run time for this example is less than 10 minutes using a generic desktop computer with single a Intel Pentium 4 processor and 1 GB RAM. The computer code for the fixed point algorithm is written in MatLab 6.5 and calls a gradient projection subroutine for which the control, state and adjoint variables are determined in the sequential fashion explained in Section 4.

A Comment on Scalability of the Fixed Point Algorithm. The run time of the fixed point algorithm reduces dramatically when the joint constraints (7) are relaxed in each firm's extremal problem. This observation is in line with the observations made in Pang [17], Harker [9] and Pang and Hobbs [18]: GNEPs are computationally more demanding. The network on which we have tested our algorithm is admittedly a simplified one, our ongoing research concentrates on the scalability of the algorithm, in particular testing on the northwest European electricity market formed by Belgium, France, Germany and the Netherlands (Neuhoff et al., [15]). The performance of the algorithm largely depends on the ability to

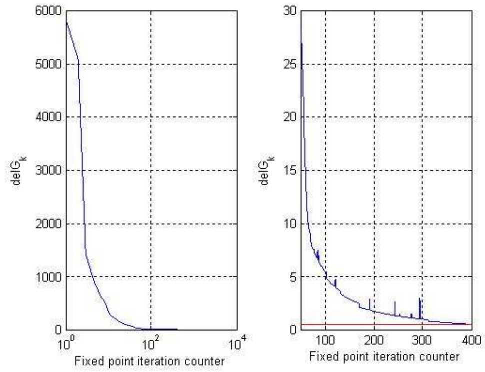

FIGURE 4. Performance of fixed point iteration : (*left*) plot of Δ_k vs. k (semi-log scale)

solve the minimum norm projection subproblem in each iteration quickly and efficiently. Special structures of the problem may be exploited based on the specific application. It is also possible, when certain structural property exists, to recast the minimum norm projection subproblem as a linear quadratic optimal control problem – a very well studied class of optimal control problems for which a closed form control law exists.

5.2.2. *Ramping Rate Trajectories.* In Figures 5, 6 and 7 we plot ramping rate trajectories of different power units owned by firm 1, 2 and 3 respectively. As a general trend, we observe that the gas turbines are ramped up and down at the full capacity (ramping capacity constraints are binding at those points) to catch up with the market demand in a short notice.

5.2.3. *Power Generation Rates.* Figures 8, 9 and 10 plot corresponding generation rate trajectories of different power units owned by firms $1, 2$ and 3 respectively. As expected, our simulated result demonstrates that in equilibirum coal units are operated at least 50% of their respective capacities throughout the day. However, the expensive gas turbines (marked as unit 2 in the figures) are only used only during peak load periods and during the steep end-of-day ramp down period, when additional ramping capability is needed. Consistent with the latter conclusion, note that the peak unit 2 output occurs after the time of highest sales, and just before the period of most rapid decrease in demand (Figure 11).

5.2.4. *Regional Sales.* We plot the total regional sales by all 3 firms at every regions over time in Figure 11. We observe that the joint sales capacity constraints were binding in market 2 and 3 right after 8 pm.

5.2.5. *Market Price.* Market prices of electricity (expressed in \$/MWh) at 3 different regions (markets) are plotted over time in Figure 12. Note that prices are highly correlated with loads (Figure 11). In this example, the price spikes often observed in California during steep ramping periods do not occur. This is due in part to the smoothness of demand changes, which is partially a result of the assumed instantaneous responsiveness of load to price changes. Higher prices during peak periods lower peak demands, and thus decrease the rate of change in demand during ramp up and down.

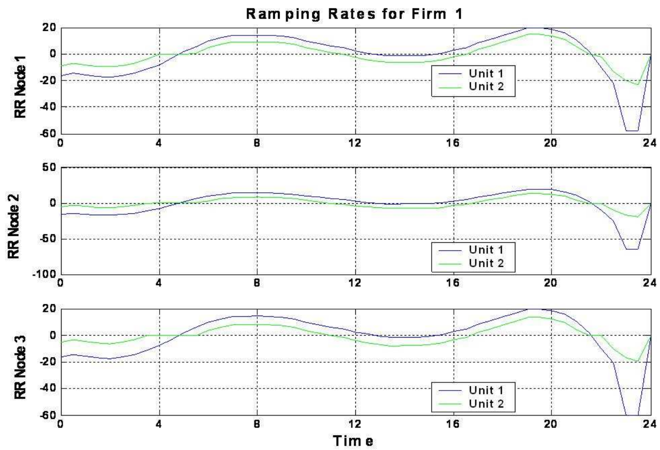

FIGURE 5. The equilibrium ramping strategies for firm 1, grouped by the nodes

6. **Concluding remarks.** We present a dynamic Cournot-Nash model of competition among power generators on a transmission network that includes two new features in each generator's profit maximization optimal control problem : (i) a set of joint constraints arising from the joint regional sales cap; (ii) explicit consideration of generators' ramping costs and constraints. These extensions have important economic and operational applications. The joint sales cap constraints induce additional analytical and computational challenges. To overcome those, we provide a restricted formulation involving partitioned differential variational inequalities. We have shown an equivalence relationship between solutions of this DVI formulation with the Cournot-Nash equilibrium of the model, and shown that under very mild regularity conditions there exists at least one equilibrium. We also discuss a version of implicit fixed point algorithm which can be employed to compute our model.

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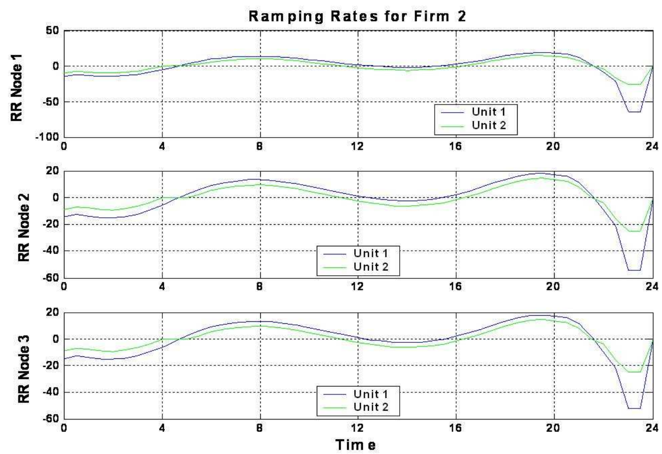

FIGURE 6. The equilibrium ramping strategies for firm 2, grouped by the nodes

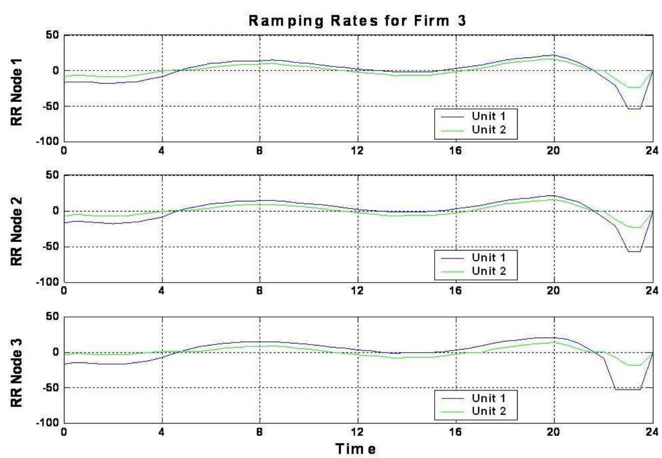

FIGURE 7. The equilibrium ramping strategies for firm 3, grouped by the nodes

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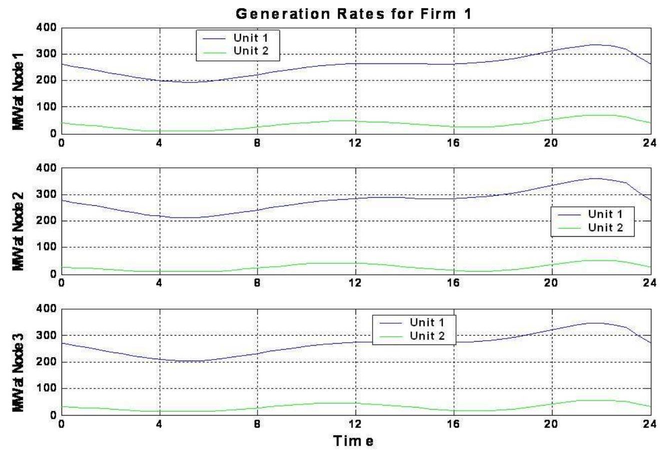

FIGURE 8. Generation rates of firm 1 over time, grouped by the location and generator type

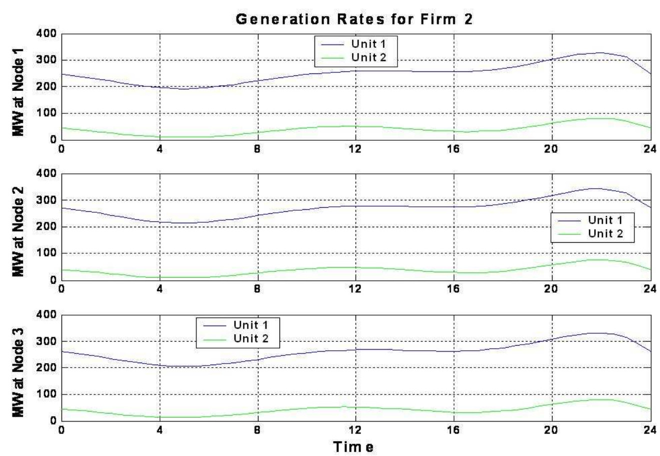

FIGURE 9. Generation rates of firm 2 over time, grouped by the location and generator type

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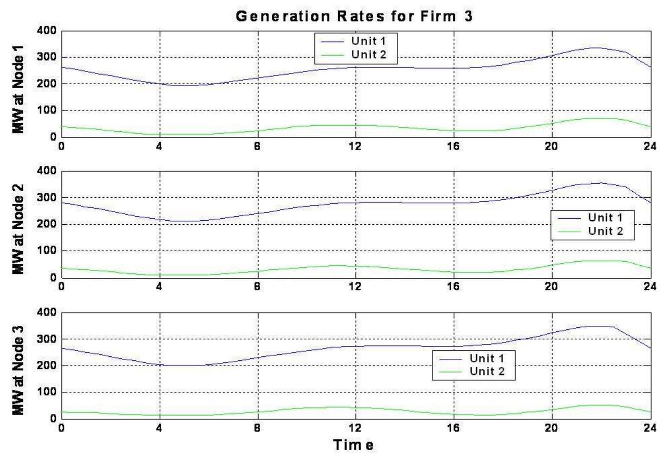

FIGURE 10. Generation rates of firm 3 over time, grouped by the location and generator type

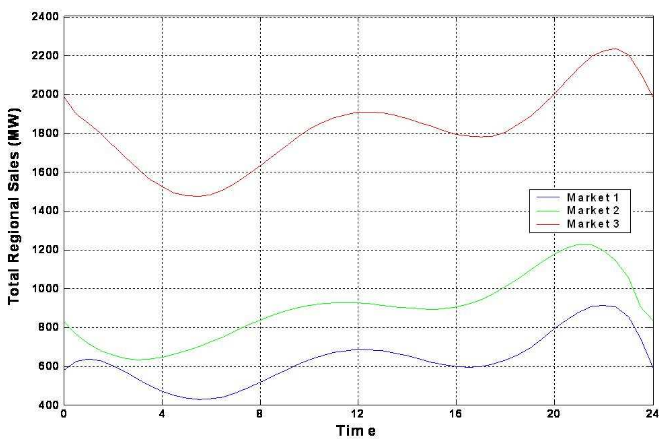

FIGURE 11. Total regional sales over time at 3 different markets (regions)

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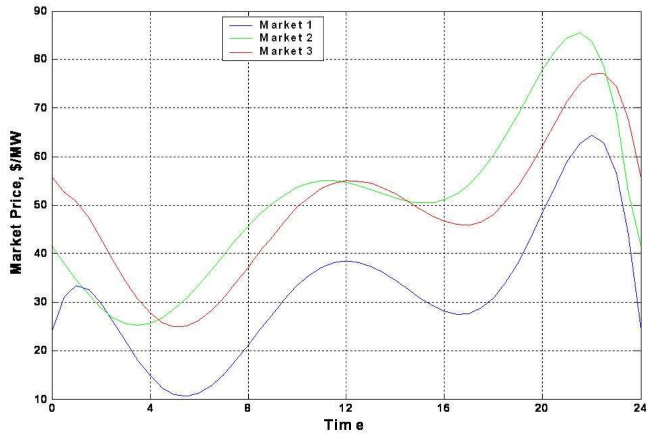

FIGURE 12. Spot market prices (at equilibrium) of power at 3 different markets (regions) at different times of a day

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