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Tacit Collusion Games in Pool-Based Electricity Markets under Transmission Constraints

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Abstract Harrington et al. [21] introduced a general framework for modeling tacit collusion in which producing firms collectively maximize the Nash bargaining objective function, subject to incentive compatibility constraints. This work extends that collusion model to the setting of a competitive pool-based electricity market operated by an independent system operator. The extension has two features. First, the locationally distinct markets in which firms compete are connected by transmission lines. Capacity limits of the transmission lines, together with the laws of physics that guide the flow of electricity, may alter firms' strategic behavior. Second, in addition to electricity power producers, other market participants, including system operators and power marketers, play important roles in a competitive electricity market. The new players are included in the model in order to better represent real-world markets, and this inclusion will impact power producers' strategic behavior as well. The resulting model is a mathematical program with equilibrium constraints (MPEC). Properties of the specific MPEC are discussed and numerical examples illustrating the impacts of transmission congestion in a collusive game are presented.

Keywords collusion · electricity market · transmission · MPEC

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1 Introduction

Since the restructuring of electricity markets in the U.S. and other countries around the world, much attention has been given to the oligopolistic nature of the newly created wholesale electricity markets. Certain features of such markets, including a relatively small number of incumbent power generating firms, barriers to entry for new plants, isolation of submarkets due to transmission congestion, inelastic demand, and non-storability of electricity, have all contributed to the ability of incumbent power generators to increase their price bids above their true marginal costs. As a result, market clearing prices (the prices at which supply meets demand) are higher than what they should be in a perfectly competitive market. Prices above competitive levels lead to income transfer from consumers to power producers. In addition, allocation and productive inefficiencies can occur in the form of too little overall production, coupled with too much coming from costly small producers and too little from cheap plants that larger producers take out of production.

Numerous market equilibrium models based upon game theory have been proposed to study oligopolistic firms' strategic interactions in deregulated electricity markets. (See Ventosa et al. [45] for a literature review.) However, most of the models focus on static interactions among firms; that is, firms are assumed to play a simultaneous-move game only once. In reality, such as in a daily electricity market, market participants interact with each other repeatedly. It is well-known in the industrial organization literature that players may behave differently in repeated games than in static games. In a static setting, players try to undercut their rivals whereas in dynamic settings, players may realize that better payoffs can be achieved if they behave more collaboratively. This leads to tacit collusion among non-cooperative players, which can lead to higher prices and more market distortion than static gaming among market participants. Rothkopf [38] expressed concern that static-game-based analysis may not predict well how actual electricity markets work. The same concern is shared by others, for example, Borenstein et al. [5], Harvey and Hogan [22], Newbery [33], and Twomey et al. [44].

Some empirical studies justified the concern that static models are not adequate to capture firms' behavior in dynamic settings. Sweeting [42] showed that between 1996 and 2000, power producers' behavior in UK's electricity market resembled tacit collusion more than static Nash equilibria. Similar results were found in Macatangay [30]. Fabra and Toro [14] found empirical evidence indicating that electricity generators may have engaged in tacit collusion during 1998 in Spain's decentralized electricity market.

Simulation-based models have also shown that repeated interactions among simulated agents can lead to tacit collusion. Such studies include Visudhiphan and Ilić [46], Bunn and Oliveira [7], Bunn and Martoccia [6], Correia et al. [11], Tellidou and Bakirtzis [43], and Anderson and Cau [1].

Equilibrium-based repeated-game models on electricity markets have been scarce. Most of the existing works employed a joint monopoly model (which is equivalent to an explicit collusion, or a cartel), such as in Fabra [13] and in

Puller [36]. Such models are better suited to study cooperative games with exchangeable utility (i.e., side-payments). Arguably, however, cooperative games with non-exchangeable utility are more appropriate for market games, because the existence of side-payments would represent a 'smoking gun' for antitrust enforcers. In contrast, tacit collusion is much more difficult to prove in court because agreements are implicit.

A canonical repeated-game model of tacit collusion was established in Harrington et al. [21], where the model was formulated as an optimization problem. Fershtman and Pakes [15] used a similar approach to model oligopolistic collusion, but the constraint set in their optimization model is much simpler than that in [21]. Rasch and Wambach [37] applied the framework in [21] to study antitrust issues. Here we present an extension of the model in [21] and considers a repeated game in a deregulated electricity market with transmission constraints. Though the network effect on collusion in the form of multimarket contact has been studied in the well-known paper by Bernheim and Whinston [3], the effect of network congestion on collusion is not well-understood in existing literature. Such an effect, however, is key to study strategic electric power suppliers' behavior in an organized wholesale electricity market. To our knowledge, this paper represents the first attempt to model such behavior in a repeated-game setting with congested transmission networks.

There is not a unique way to include transmission in the tacit collusion framework of Harrington et al. [21], and transmission poses challenges for computation and analysis. One fundamental choice concerns the organization of the transmission market. On one hand, transmission markets can be operated separately from generation markets, so that generators wishing to sell in other markets would have to buy transmission capacity separately from selling energy. For instance, this was the structure in northwestern Europe until recently, where, e.g., power producers in Germany had to buy transmission capacity to the Netherlands in one auction and sell their power in another. Local market rules, for example concerning curtailment of transactions in case of line outages, could also hamper access by foreign competitors or traders. The lack of explicit coordination of generation and transmission auctions resulted in inefficiencies, and could present opportunities for generators in different countries to tacitly collude through geographically market sharing arrangements. The other possible transmission market organization is a pool market operated by an Independent System Operator (ISO), in which generators sell their output to the system operator at their location. The ISO then transmits the power, and electricity prices are set based on marginal production costs and congestion. In such markets, generators can also sell power in other markets by paying the ISO a fee (transmission charge), which is defined as the electricity price differential between two markets. Thus, by definition, the ISO perfectly arbitrages price differences over space. As a consequence, the generator will be indifferent to selling locally or scheduling a sale elsewhere. In this paper, we focus on this second type of transmission organization, which characterizes the major organized power markets in the U.S.

Given that focus, the second fundamental transmission modeling choice pertains to generators' expectations about the interaction of their output decisions with the ISO's decisions concerning transmission congestion and prices. Our framework requires that three distinct sets of market solutions be calculated, and so this choice must be made for each of these sets. These sets are: (1) the collusive solution itself that the generators tacitly agree to (based on the Nash bargaining solution); (2) the optimal unilateral deviation from that agreement that each player would pursue in the short term if decides to abandon collusion; and (3) the Nash noncooperative solution, which is assumed to occur in the long term if the collusive agreement breaks down. The Nash bargaining solution is found by maximizing a function of the generator profits subject to the incentive compatibility constraint that each player cannot increase its present worth of profits by unilaterally deviating in the short term followed by the entire market reverting to the Nash noncooperative solution. For each of those solutions, the modeler must decide how to represent the relationship of the generators to the transmission independent system operator (ISO). Sophisticated generators might anticipate how their decisions would affect the ISO's charges for transmission services; less sophisticated generators might assume that the charges will not change; and the least sophisticated assumption is that any changes in generation are absorbed by local demand and do not affect transmission flows. Each representation has advantages and disadvantages relative to market realism as well as tractability. The model in this paper is based upon one set of transmission modeling choices for the three solutions. As we point out below, others are possible, and would likely yield different solutions. The contribution of this paper is its description of possible ways to model transmission-constrained tacit collusion in ISO markets, and the presentation and analysis of one particular implementation which we believe represents a reasonable compromise between modeling realism and computational feasibility.

The organization of the rest of the paper is as follows. Section 2 discusses the setting of a deregulated electricity market that we model. Section 3 presents the collusion model formulation. Reformulations of the collusion model into computationally convenient forms and corresponding model properties are provided in Section 4. Numerical examples illustrating the effects of transmission constraints on collusive firms' strategies are shown in Section 5. Section 6 summarizes the results and discusses future research.

2 Electricity Market Setting

In a deregulated electricity market, power producers bid into a spot market to sell electricity each day, with the spot market operated by an independent system operator (referred to as the ISO). The ISO dispatches electricity through the transmission network to balance supply and demand in order to maximize the net benefits to the market (benefits to consumers minus costs to producers, as revealed by their bids). Market clearing prices are calculated by the ISO for each location and time period based on the dual variable of the energy balance at each node (or 'bus') in the network. A spot wholesale power market may consist of a two-settlement system – a day-ahead (DA) market and a real-time (RT) market. In such a system, both electricity supply and demand bids for the next day are collected in the DA market, and the ISO selects generators to run in the next day to meet the projected demand. In the RT market, the ISO dispatches based on the day-ahead schedules, and deploys additional resources to cover any real time disparity between supply and demand. Uncertainties, such as errors in demand forecasts, variable outputs from intermittent renewable resources, and forced outages of generators or transmission lines, play an important role in analyzing generators' behavior in real-time. Since considering uncertainty is beyond the scope of this work, the collusion model to be introduced can be viewed as a simplified representation of a DA market in which forecasted generation availability and demand are known to all market participants. As a result, we consider a repeated game resembling the daily repetition of a day-ahead market, which includes market-power exercising producers, an ISO, and price-taking consumers.

3 Collusion Model Formulations

3.1 Collusion Model without Transmission Constraints

To make this paper self-contained, we first briefly introduce the generic collusion model, as first presented in Harrington et al. [21]. Starting with a static game, we consider its normal-form representation, which specifies the number of players, their action spaces, and payoff functions. A repeated game consists of repetition of a static game (also referred to as a single-stage game) whose normal form does not change over time. We consider a simultaneous-move single-stage game, and assume that it is of complete information; that is, each player's payoff function is common knowledge to all the players.

Consider a single-stage game with F players. Let $\mathcal{F} = \{1, \ldots, F\}$ denote the set of players, and $X_f \in \Re^{n_f}$ denote the feasible action space of player f. We call the Cartesian product of the individual action spaces, $X := \prod_{f \in \mathcal{F}} X_f$, the action space of the single-stage game. Each player has a payoff function $\pi_f(x)$ which maps from the game's action space X (instead of from the player's own action space X_f) to a value in $\Re$. Let π denote the vector of all players' payoffs; that is, $\pi := (\pi_1, \ldots, \pi_F)$. Then the normal-form representation of the single-stage game is denoted by the triplet $(\mathcal{F}, X, \pi)$.

Assume that players are rational; that is, each player maximizes his or her own payoff. Player f's payoff maximization problem, parameterized by other firms' actions x_{-f}, is as follows.

$$\begin{aligned} \underset{x_f}{\text{maximize}} \;\; & \pi_f(x_f, x_{-f}) \\ \text{subject to} \;\; & x_f \in X_f. \end{aligned} \tag{1}$$

A combination of players' strategies, $x^N = (x_1^N, \ldots, x_F^N) \in X$, is referred to as a (pure-strategy) Nash equilibrium to the single-stage game $(\mathcal{F}, X, \pi)$ if

$$\pi_f(x_f^N, x_{-f}^N) \geq \pi_f(x_f, x_{-f}^N), \;\forall x_f \in X_f, \; f \in \mathcal{F}.$$

In an infinite repetition of a single-stage game, it is well-known by the so-called 'folk theorem' (Friedman [17]) in game theory that many Nash equilibria can exist with a given discount factor $\delta \in (0,1)$,[1] and some may lead to a higher payoff to each player than their payoff in a static equilibrium.

One strategy that can yield higher equilibrium payoffs is the so-called grim-trigger strategy (Friedman [17]). Suppose that players compete to supply a homogeneous good, and let q_f denote the quantities of the good that player f chooses to supply. Assume that a Nash equilibrium exists in a single-stage game, denoted by $q^N = (q_1^N, \ldots, q_F^N)$. Suppose that there exists $\tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_F) \in X = \prod_{f \in \mathcal{F}} X_f$ such that $\pi_f(\tilde{q}) > \pi_f^N$ for every $f \in \mathcal{F}$. Then a grim-trigger strategy in an infinitely repeated game, with time period starting from 1 and indexed by t, is as follows.

$$
\begin{aligned}
& q_f^1 = \tilde{q}_f \\
& q_f^t = \begin{cases} \tilde{q}_f & \text{if } q^\tau = \tilde{q} \quad \forall \tau \in \{1, 2, \ldots, t-1\} \\ q_f^N & \text{otherwise.} \end{cases} \qquad \forall t \in \{2, 3, \ldots\}.
\end{aligned} \tag{2}
$$

$(\tilde{q}, q^N)$ is referred to as a grim-trigger strategy combination, which says that players choose to supply $\tilde{q}$ unless one or more firms deviate from $\tilde{q}$ in a certain t. Assume that the repeated game is of perfect information; that is, the entire history of the game is known to all players. Then deviation at one period is observed by all firms in the next period, which triggers the punishment in which all firms choose q^N throughout the remaining repeated game. Friedman [18] presented a condition for a grim-trigger strategy to be not only a Nash equilibrium, but also a subgame-perfect equilibrium (SPE) in a repeated game – a refinement of a Nash equilibrium. An SPE is a Nash equilibrium to each subgame of a repeated game, where a subgame is a game that starts from t and contains all the stage games onwards, with $t = 1, 2, \ldots,$.[2]

Theorem 1 *(Friedman [18]) Let an infinitely repeated game consist of a repetition of a simultaneous-move stage game $G := (\mathcal{F}, X, \pi)$. Let $q^N = (q_1^N, \ldots, q_F^N) \in X$ be a Nash equilibrium of the stage game G, and let $(\tilde{q}, q^N) \in X \times X$ be a grim-trigger strategy combination. Then $(\tilde{q}, q^N)$ is a subgame-perfect equilibrium of the infinitely repeated game if and only if*

$$
\delta \geq \frac{\pi_f^d(\tilde{q}_{-f}) - \pi_f(\tilde{q})}{\pi_f^d(\tilde{q}_{-f}) - \pi_f^N}, \ \forall f \in \mathcal{F}, \tag{3}
$$

where $\pi_f^d(\tilde{q}_{-f})$ is firm f's payoff from its best response to other firms' action $\tilde{q}_{-f}$; that is,

$$
\pi_f^d(\tilde{q}_{-f}) \equiv \left\{ \begin{array}{l} \underset{q_f}{\text{maximize}} \ \pi_f(q_f; \tilde{q}_{-f}) \\ \text{subject to } q_f \in X_f. \end{array} \right\} \tag{4}
$$

□

[1] The discount factor can be defined as $\delta := (1-p)/(1+r)$, where $r \in [0,1]$ is an interest rate and $p \in [0,1]$ represents a probability that the repeated game will end in the next time period.

[2] By definition, the repetition of a static Nash equilibrium is also an SPE. However, not any Nash equilibrium of a repeated game is an SPE.

We can re-write the formula in (3) as follows (dropping the tilde of q):

$$\pi_f(q) \ \geq \ (\, 1 - \delta\,)\, \pi_f^d(q_{-f}) + \delta\, \pi_f^N, \ \forall f \in \mathcal{F}, \tag{5}$$

and refer to this as player f's incentive compatibility constraint. The interpretation is that the overall gain of one-time cheating is no more than the overall payoff of maintaining collusion. Hence, it is each payoff-maximizing player's self-interest to choose q_f instead of cheating. We then define the set

$$\Omega_\delta \ \equiv \ \{q \in X = \prod_{f \in \mathcal{F}} X_f : \ \pi_f(q) \geq (1-\delta)\pi_f^d(q_{-f}) + \delta\pi_f^N, \ \forall f \in \mathcal{F}\}. \tag{6}$$

The set Ω_δ includes the feasible subgame-perfect quantities leading to collusive profits that provide no incentive to any firm to deviate. This set (plus any technical constraints upon the quantities) defines the feasible region for the collusive model.

The set Ω_δ is not a singleton in general as there may be multiple subgame-perfect equilibria. When facing multiple equilibria, players need to have a mechanism to select a single equilibrium. One of such mechanisms is through bargaining. To be more specific, as players can achieve higher payoffs in a repeated game than in a static game, they decide the actual allocation of the surplus by bargaining with each other. There is a possibility that no agreement is reached and the bargaining breaks down, with each player receiving the payoff in a static Nash equilibrium.

Instead of explicitly modeling the bargaining process, the axiomatic bargaining theory developed by Nash [32] is employed in this work. Nash [32] listed four axioms to be satisfied by an equilibrium from a bargaining process with nonexchangeable utility, and showed that the bargaining problem has a unique solution satisfying the axioms if and only if the payoffs to the players are as follows.

$$\pi^* := (\pi_1^*, \ldots, \pi_F^*) \in \arg\max_{\pi \in \Gamma} \prod_{f=1}^{F} (\pi_f - \nu_f), \tag{7}$$

where $\nu = (\nu_1, \ldots, \nu_F)$ with ν_f being player f's payoff should the bargaining fail, and $\Gamma = \prod_{f=1}^{F} \Gamma_f$ with Γ_f being the set of player f's feasible payoffs. The justifications for using the Nash bargaining approach to select a subgame-perfect equilibrium can be found in Harrington [20]. The complete model that uses the Nash bargaining framework as a mechanism to select an equilibrium from the set of feasible incentive-compatible quantities (Ω_δ) is as follows.

$$\begin{aligned} &\underset{q}{\text{maximize}} \ \Theta(q) \equiv \prod_{f \in \mathcal{F}} [\pi_f(q) - \pi_f^N] \\ &\text{subject to } q \in \Omega_\delta \end{aligned} \tag{8}$$

The implicit optimal value functions $\pi_f^d(q_{-f})$ in the set Ω_δ pose difficulties in solving (8) as they do not have explicit function forms. However, the following proposition shows that $\pi_f^d(q_{-f})$ can be explicitly represented by introducing auxiliary variables, and the collusion model can be reformulated as a mathematical program with equilibrium constraints (MPEC).

Proposition 1 *(Liu [28]) For each player $f \in \mathcal{F}$, suppose its feasible action space X_f can be explicitly represented as follows: $X_f = \{q_f \in \Re^n : g_f(q_f) \leq 0\}$, where $g_f : \Re^n \to \Re^m$ is assumed to be convex and continuously differentiable. Further assume that a constraint qualification holds for X_f. Suppose that $\pi_f(q_f, q_{-f})$ is concave and continuously differentiable, both in regard to q_f. Then model (8) is equivalent to the following MPEC.*

$$\begin{aligned}
&\underset{q,\ q^*,\ \gamma^*}{\text{maximize}}\ \Theta(q) = \prod_{f \in \mathcal{F}} [\pi_f(q) - \pi_f^N] \\
&\text{subject to } \pi_f(q) \geq (1-\delta)\pi_f(q_f^*, q_{-f}) + \delta \pi_f^N,\ \forall f \in \mathcal{F} \\
&\qquad\qquad g_f(q_f) \leq 0,\ \forall f \in \mathcal{F}, \\
&\qquad\qquad -\nabla_{q_f^*}\pi_f(q_f^*, q_{-f}) + \nabla g_f(q_f^*)^T \gamma_f^* = 0,\ \forall f \in \mathcal{F} \\
&\qquad\qquad 0 \leq \gamma_f^* \perp -g_f(q_f^*) \geq 0,\ \forall f \in \mathcal{F}.
\end{aligned} \tag{9}$$

□

3.2 Tacit Collusion Model with Transmission Constraints

To extend the collusion model from the previous section to the setting of an electricity market, we need to model not only different market participants as described in Section 2, but also the transmission network structure. Various model formulations based on different assumptions concerning market rules, as well as the resulting level of computational difficulty, are discussed in this subsection. In particular, in each of three market submodels that make up the overall tacit collusion model, it is necessary to make an assumption about whether or not the ISO's reaction to changes in generator decisions are explicitly considered by the generators. The three submodels are those of power prices when firms collude; when a single firm deviates from collusion; and the punishment stage when firms revert to a Nash equilibrium. In theory there are 2^3 possible combinations of transmission assumptions. We then settle on one that we believe represents a reasonable compromise between computational tractability and realism.

To ease presentation, we first summarize the notation to be used by the models in this and later sections.

Sets, Indices and Dimensions

$\mathcal{N}$	Set of nodes in a network; $\lvert\mathcal{N}\rvert = N$. $i, j \in \mathcal{N}$ means that node i and j are in $\mathcal{N}$
$\mathcal{A}$	Set of (directional) links in the transmission network $\mathcal{N}$; $\lvert\mathcal{A}\rvert = 2L$
$\mathcal{F}$	Set of power producers; $\lvert\mathcal{F}\rvert = F$. $f \in \mathcal{F}$ means that power producer f is in $\mathcal{F}$.

Parameters

P_i^0	Price intercept of the affine inverse demand function at node i [$/MWh]
Q_i^0	Quantity intercept of the affine inverse demand function at node i [$/MWh]
$C_{fi}(x)$	Firm f's production cost function at node i [$/hr]
PTDF_{ki}	(k, i)-th element of the power transmission-distribution factor matrix
T_k	Transmission capacity on link $k \in \mathcal{A}$ [MW].

Variables

g_{fi}	Firm f's electricity generation (in a collusive solution) at node $i \in \mathcal{N}$ [MWh]
y_i	Power transmitted from the hub to node i in linearized DC approximation [MW/hr]
w_i	Fees charged for transmitting electricity from the hub to node $i \in \mathcal{N}$ [$/MWh]
d_i	Electricity demand at node $i \in \mathcal{N}$ [MWh]
p_i	Power price at node $i \in \mathcal{N}$ [$/MWh].

Vectors and Matrices

$\mathbf{1}$	A vector of all 1's with a proper dimension
I	An identity matrix with a proper dimension
E	A square matrix of all 1's with a proper dimension
P^0	$N \times 1$ vector of P_i^0's for with $i \in \mathcal{N}$
B	$N \times N$ diagonal matrix with the i-th diagonal entry being P_i^0/Q_i^0
g	$F \times N$ matrix with the (f,i)-th entry being g_{fi}
G, G_{-f}	$N \times 1$ vectors, with $G_i = \sum_{f \in \mathcal{F}} g_{fi}$ and $G_{-f_i} = \sum_{f \neq h \in \mathcal{F}} g_{hi}$
y	$N \times 1$ vector of of y_i's, $i \in \mathcal{N}$
T	$2L \times 1$ vector of T_k's, the transmission line capacities, $k \in \mathcal{A}$.

There are two key elements in building the collusion model within the context of a deregulated electricity market: (1) the single-stage game to be repeated among various market players; and (2) the optimal payoff when a firm unilaterally deviates from collusion. These two elements are discussed in detail in the following subsections.

3.2.1 Static Games to be Repeated

In a single-stage game, power producers are assumed to play a simultaneous-move Cournot game. The justification for the simultaneous-move game is that when they make generation decisions (in a time epoch), power producers do not know their rivals' actions (in the same epoch). However, whether the power producers anticipate their actions to the ISO's dispatch decisions depends on market structures and participants' rationality. Before presenting the possible formulations of producers' optimization problems, we first present the ISO's problem and the corresponding transmission network modeling.

In this work we consider a hub-spoke type of network representation, in which the electricity flow from node i to j is assumed to be from i to a hub, and from the hub to j.[3] By doing so, the number of variables representing electricity flows can be significantly reduced. We also consider a lossless transmission network throughout the work.[4]

[3] This is equivalent to the linearized DC representation of power flow, which is an approximation of the actual AC load flow (see Schweppe et al. [41]) and is commonly used in models of electricity markets [45]. In that representation, a hub is arbitrarily chosen, and PTDFs represent the flow on a particular transmission element resulting from a unit injection at the hub and a withdrawal at some other node i. Linearity implies that a transfer of power from a node j to a node i can be modeled (and priced) as two transactions: from j to an arbitrary hub, and then from that hub to i.

[4] Chen et al. [9] have proposed a DC load flow model that considers transmission losses through a (convex) quadratic function. Such an implementation can be incorporated into the modeling and computational approach of the collusion model without much difficulty. As a starting point and for the ease of argument, we omit losses.

The ISO collects the bids from power producers and dispatches to maximize the social surplus. Let y_i denote the dispatch decision of the ISO, with a positive y_i meaning sending power to node i (from the hub node), and a negative y_i meaning withdrawing power from node i (to send to the hub node). As a result, the demand of electricity at a node (d_i) is met by $G_i + y_i$, the total generation at i plus the power shipped to (or withdrawn from) i.[5] Then the ISO's social-surplus-maximization problem is given as follows. (The same formulation has been used in several previous works, including [47,48].)

$$\begin{aligned}
\underset{y}{\text{maximize}} \quad & \sum_{i\in\mathcal{N}} \left[\int_0^{G_i+y_i} p_i(\tau_i)d\tau_i - \sum_{f\in\mathcal{F}} C_{fi}(g_{fi}) \right] \\
\text{subject to} \quad & \sum_{i\in\mathcal{N}} y_i = 0 \quad (\mu) \\
& \sum_{i\in\mathcal{N}} \text{PTDF}_{ki} y_i \le T_k, \ \forall k \in \mathcal{A}. \quad (\lambda_k)
\end{aligned} \tag{10}$$

The objective function in (10) is the classic definition of social surplus, which equals the area under the inverse demand function minus the total generation cost. The first constraint in (10) is a flow balance constraint; while the second is the transmission capacity limits, with the Kirchhoff Current and Voltage Laws implicit in the power transmission and distribution factor matrix (PTDF). The greek letters in the parenthesis represent the corresponding Lagrangian multipliers. The interpretation of μ is the electricity price at the hub node; while the prices at other nodes can be obtained by writing out the optimality conditions of (10):

$$p_i(G_i + y_i) = \mu + \sum_{k\in\mathcal{A}} \text{PTDF}_{ki}\lambda_k, \tag{11}$$

which are exactly the locational marginal prices (Hogan [25]) at each node. With the formulation (11), it can be seen that in an optimal dispatch schedule, the price differentials at two different nodes are only caused by transmission congestion rents (allocated to each node through the power transmission and distribution factors);[6] that is,

$$p_i - p_j = \sum_{k\in\mathcal{A}} \text{PTDF}_{ki}\lambda_k - \sum_{k\in\mathcal{A}} \text{PTDF}_{kj}\lambda_k, \ \forall i,j \in \mathcal{N}. \tag{12}$$

The game played between power producers and the ISO can be of either of two types.[7] If power producers believe that their actions would not affect the

[5] Note that since we use an affine function to represent electricity demand at each node, instead of using a fixed demand, there will not be the case that the supply cannot meet the demand.

[6] This is essentially the ISO's role is, in essence, to eliminate non-congestion related price differentials. Due to this role of the ISO, there cannot be "market sharing" type of agreements (Belleflamme and Bloch [2]) between power producers to preserve their monopoly positions in their "home" markets.

[7] There is another possible type of games (Sauma and Oren [40]) in which the ISO plays as the leader while the oligopolistic power producers act as the follower, with respect to the dispatch schedules. The rationale is that the ISO is aware of the market imperfection, and

ISO's dispatch, then the producers do not consider the ISO's actions endogenously. This is sometimes referred to as Bertrand competition between power producers and the ISO (as in Hobbs [23] and Metzler et al. [31]). The other type is that power producers anticipate the impacts of their actions on the transmission network and endogenously consider the ISO's dispatch problem (as in Cardell et al. [8], Hobbs et al. [24] and Yao et al. [47]). The difference between the two types of models is illustrated in Figure 1.

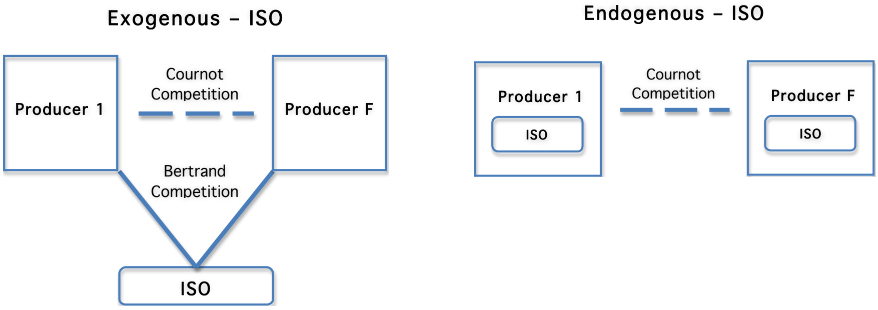

Fig. 1: Comparison of the exogenous-ISO model and endogenous-ISO.

We adopt the assumption corresponding to the endogenous-ISO model for power producers' rationality when they engage in a collusive game, as it is not unreasonable to expect that collusive firms are expected to be sophisticated in terms of the knowledge of the game. The corresponding collusive-game structure is illustrated in Figure 2.

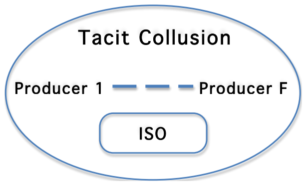

Fig. 2: Illustration of the collusion model in an electricity market.

For all the models discussed below, we also assume that consumers' willingness-to-pay at each location within the network is represented by an affine inverse demand function. That is, the electricity price $p_i(d_i)$ at location $i \in \mathcal{N}$, which is a function of total demand at i (denoted by d_i), is determined as

$$p_i(d_i) = P_i^0 - \frac{P_i^0}{Q_i^0} d_i, \tag{13}$$

where P_i^0 and Q_i^0 are the intercepts of the affine function at the price and quantity axis, respectively. With the above-discussed assumptions and by following

mitigates the potential market power abuse by strategic generators through anticipating their actions. Under such a setting, the ability that generators may form a tacit collusion is expected to be limited. Such an interesting direction is left to be explored in future research.

the framework introduced in Section 3.1, we establish the collusive-game model of an electricity market as follows.

$$\begin{aligned}
&\underset{g,\ a,\ y,\ w,\ p}{\text{maximize}}\ \Theta(g,a,y) = \prod_{f\in\mathcal{F}} \left[\pi_f(g) - \pi_f^N\right] \\
&\text{subject to}\ \pi_f(g) \geq (1-\delta)\pi_f^d(G_{-f}) + \delta\pi_f^N,\ \forall f \in \mathcal{F} \\
&\qquad\qquad g_{fi} \in X_{fi},\ \forall f \in \mathcal{F},\ i \in \mathcal{N} \\
&\qquad\qquad \text{ISO}\left\{\begin{array}{l}
\underset{y}{\text{maximize}}\ \displaystyle\sum_{i\in\mathcal{N}} \left[\int_0^{y_i+G_i} p_i(\tau_i)d\tau_i - \sum_{f\in\mathcal{F}} C_{fi}(g_{fi})\right] \\
\text{subject to}\ \displaystyle\sum_{i\in\mathcal{N}} y_i = 0 \\
\qquad\qquad\ \displaystyle\sum_{i\in\mathcal{N}} \text{PTDF}_{ki} y_i \leq T_k,\ \forall k \in \mathcal{A}.
\end{array}\right\}.
\end{aligned} \tag{14}$$

The set X_{fi} in (14) represents the feasible production set of firm f at node i. X_f then is defined as the Cartesian product of X_{fi} for $i \in \mathcal{N}$. We make the blanket assumption throughout that X_{fi} is nonempty, compact and convex. Note that the formulation above models a Poolco-type electricity market in which generators bid all their generation at the location that the electricity is produced and the ISO dispatches to balance the supply and demand of the entire network. The model can also be modified to model a bilateral market, or a hybrid market (as in Hobbs [23] and Metzler et al. [31]), where generators can either bid to the ISO or sign bilateral contracts with a buyer (i.e., self-scheduling). Since the ISO dispatches electricity to ensure that the price differential between two locations are only caused by transmission charges when congestion occurs, generators earn the same revenue whether they sell at the point of generation or elsewhere. Hence, assuming selling at the point of generation will not limit the model in any essential way (Metzler et al. [31]).

The optimization problem (14) is not convex in general. It is a bilevel programming problem, and the presence of transmission constraints in ISO's problem renders it an MPEC. There are further modeling and computational difficulties associated with the formulation in (14). First, the firm's optimal deviation profit, $\pi_f^d(G_{-f})$, becomes more complicated due to the added market players and the network constraints. The detailed formulation of $\pi_f^d(G_{-f})$ is the focus of the following subsection. Second, the static Nash equilibrium π_f^N is not unambiguously defined. To be consistent with the assumption that power producers endogenously consider the ISO's problem in a collusive game, π_f^N should be the payoff in a static Nash equilibrium corresponding to the endogenous-ISO model, as illustrated by the right-hand picture in Figure 1. However, in such an equilibrium, due to the transmission constraints, each firm's profit maximization problem is an MPEC. The overall model then becomes an equilibrium problem with equilibrium constraints (EPEC). Such models have been discussed in detail in [27, 26, 47], and are widely known to have two basic issues. First, an equilibrium may not exist (Pang and Fukushima [35]). Second, even if an equilibrium exists, they are difficult to compute (heuristic algorithms have been proposed in Hu and Ralph [26] and Yao et al. [47], for example). On the other hand, the equilibrium corresponding to the exogenous-ISO assumption (the left-hand picture in Figure 1) can be formulated as a complementarity problem (CP) instead of an EPEC, and possesses desired properties (such as existence of a solution and uniqueness of firms' profits). Well-established algorithms for complementarity problems, such as PATH [12], can also be easily applied to compute an equilibrium (Metzler et al. [31]).

Therefore, to balance computational tractability and model consistency, we assume that when a deviation is detected in a repeated game, the single-stage game repeated in the punishment stage is the complementarity-problem-based Cournot game as in Metzler et al. [31].[8] Let π_f^{EPEC} denote firm f's profit in an equilibrium corresponding to the EPEC model (assuming that an equilibrium exists), and let π_f^{CP} denote firm f's profit in a single-stage equilibrium derived from the complementarity-problem-based model. It is not unreasonable to expect that $\pi_f^{EPEC} > \pi_f^{CP}$ when the input data are the same, as firms are more sophisticated in the EPEC model.[9] If this is indeed true, then using π_f^{CP} to replace π_f^{EPEC} in the punishment stage may help sustain a collusion as the punishment to all firms is more severe. If the reverse is true, namely, $\pi_f^{EPEC} < \pi_f^{CP}$, then using π_f^{CP} as the payoffs in the punishment stages will make collusion more difficult. In the following discussion we still use π_f^N (instead of specifying π_f^{EPEC} or π_f^{CP}) to denote a generic single-stage Nash equilibrium payoff in the punishment stage, and will distinguish the two different static equilibria when necessary. For completeness, the detailed formulations of the EPEC and CP model are provided in Appendix A.

3.2.2 Optimal Deviation from Collusion

Now we focus on the formulation of a firm's optimal deviation payoff π_f^d. To be consistent with the assumptions for the single-stage game, a deviating power producer should play a Stackelberg game with respect to the ISO, while taking other power producers' generation quantities, G_{-f}, as fixed. The resulting formulation is in (15).[10]

---

[8] Note that the payoffs from a static Nash equilibrium are used as inputs to the collusion model (14) and can be calculated offline. Hence, should there be theoretical and computational advances to address EPEC problems, we can easily substitute the EPEC formulation in calculating the π_f^N for each firm $f \in \mathcal{F}$.

[9] However, no proof is known to date for this claim. As a matter of fact, the EPEC game in which each power producer competes while explicitly including in its constraint set all transmission constraints and rivals generation is a generalized Nash equilibrium (GNE) (Yao et al. [47]). Such games are widely recognized to have multiple equilibria (e.g., Oren [34]). A simple example of such a GNE is the pie-sharing game, where each player chooses how much of a pie to take, given how much others have taken; it turns out that if utility is monotonic in the amount of pie, then any split of the pie among the players is an equilibrium. Consequently, there may be some equilibria in the EPEC GNE in which some of the players may be worse-off than playing the CP game.

[10] Note that the variable y_f^d with an index f does not mean that it is the deviating firm f's decision variable. The index only indicates that it is associated with the ISO's dispatch decisions when firm f deviates from the collusive solution.

$$\pi_f^d(G_{-f}) = \left\{ \begin{array}{l} \underset{g_f^d}{\text{maximize}} \displaystyle\sum_{i\in\mathcal{N}} \left[P_i^0 - \frac{P_i^0}{Q_i^0}(G_{-fi} + g_{fi}^d + y_{fi}^d) \right] g_{fi}^d - \sum_{i\in\mathcal{N}} C_{fi}(g_{fi}^d) \\ \text{subject to } g_{fi}^d \in X_{fi},\ \forall i \in \mathcal{N}, \\ \qquad \text{ISO} \left\{ \begin{array}{l} \underset{y_f^d}{\text{maximize}} \displaystyle\sum_{i\in\mathcal{N}} \left[\int_0^{y_{fi}^d + (G_{-fi}+g_{fi}^d)} p_i(\tau_i) d\tau_i \right. \\ \qquad\qquad \left. - \displaystyle\sum_{h\in\mathcal{F}} C_{hi}(g_{hi}) \right] \\ \text{subject to } \displaystyle\sum_{i\in\mathcal{N}} y_{fi}^d = 0 \\ \qquad \displaystyle\sum_{i\in\mathcal{N}} \text{PTDF}_{ki} y_{fi}^d \le T_k,\ \forall k \in \mathcal{A}. \end{array} \right. \end{array} \right\}. \quad (15)$$

With the formulation (15), $\pi_f^d(G_{-f})$ is an optimal value function of a bilevel programming problem. Under the assumption of affine price functions, the ISO's problem is a convex problem with linear constraints. As a result, the bilevel program is equivalent to an MPEC, with the complementarity constraints coming from the KKT systems of the ISO's optimization problem. This MPEC formulation incurs two theoretical difficulties. First, convexity of $\pi_f^d(G_{-f})$ is difficult to establish. Existing convexity results on optimal value functions, such as those in Fiacco and Kyparisis [16], cannot be applied to problem (15) because the objective function is not jointly concave with respect to (g_f^d, y_f^d), and the parameterized feasible region is not convex. The lack of convexity would pose both theoretical and computational difficulties for the collusion model (14). Second, the smooth reformulation given in Proposition 1, which replaces the implicit optimal value function with its KKT-system, cannot be applied to $\pi_f^d(G_{-f})$, as the feasible region in (15) is not convex. Hence the first-order condition is not sufficient to imply global optimality of the deviating firm's unilateral profit maximization.

To balance the validity of a model with computational tractability, we consider removing the ISO's problem, and let $\tilde{\pi}_f^d$ denote the resulting optimal deviating function:

$$\tilde{\pi}_f^d(G_{-f}, y) \equiv \left\{ \underset{g_f^d:\ g_f^d \in X_f}{\text{maximize}} \sum_{i\in\mathcal{N}} \left[P_i^0 - \frac{P_i^0}{Q_i^0}(G_{-fi} + g_{fi}^d + y_i) \right] g_{fi}^d - \sum_{i\in\mathcal{N}} C_{fi}(g_{fi}^d) \right\}, \quad (16)$$

which is parameterized by both other firms' generation quantity G_{-f} and ISO's dispatch schedule y under the situation where all firms collude. The underlying assumption is that the deviating firm does not believe its action would affect the ISO's dispatch decisions (y), given that the other firms still choose their collusive generation quantities (as they do not observe the deviation in the current period). In a sense, the assumption is that any change in output at one of the firm's plants is absorbed by local demand (since the dispatch is not changed). A favorable consequence of this assumption is that transmission constraints are always met when a firm deviates. Also note that $\tilde{\pi}^d(G_{-f}, y)$ is not necessarily an upper bound to $\pi^d(G_{-f})$, as the latter optimizes without the variable y being fixed. Properties of $\tilde{\pi}^d_f(G_{-f}, y)$ as well as of the collusion model (14) are discussed in the next section.

The assumption made above may be justifiable when a power producer faces high local demand, which would provide it a large one-time deviation profit. Such an assumption is nonetheless difficult to justify in a more general setting. One would expect that when a firm deviates, it would increase its output and hence lower the electricity price in its local market. Consequently, the ISO would dispatch electricity from the "local" market to other nodes where electricity prices are higher, as long as there are transmission capacities. Doing so is like to increase demand at the "local" market, and hence, would increase the "local" electricity price, making the deviating firm even more profitable. As a result, by anticipating the ISO's changes in dispatch, the deviating firm may be better off than without doing so; mathematically speaking, $\pi^d(G_{-f}) > \tilde{\pi}^d_f(G_{-f}, y)$. In this case, replacing $\pi^d(G_{-f})$ with $\tilde{\pi}^d_f(G_{-f}, y)$ would make collusion easier to sustain as the one-time deviation profit is lower. It is unclear however if the reverse relationship, $\pi^d(G_{-f}) < \tilde{\pi}^d_f(G_{-f}, y)$, could be true under certain circumstances. As a result, our work represents the first step towards modeling firms' collusive behavior under a transmission network, but is far from being complete.

4 Model Reformulation and Properties

4.1 Model Reformulation

To ease discussion of the model's properties, we first reformulate the collusion model (14) as an MPEC by writing out the optimality conditions of the ISO's problem explicitly. Let $\mu \in \Re^1$ and $\lambda \in \Re^{2L}$ denote the multipliers associated with the first and second set of constraints in the ISO's optimization problem, respectively. Under the assumption of affine inverse demand functions at each demand node and using the relaxed optimal deviating profit function $\tilde{\pi}^d_f(G_{-f}, y)$, model (14), expressed in vector form, is as follows.

$$\begin{aligned}
\underset{g,\ y}{\text{maximize}}\ & \Theta(g,y) \equiv \prod_{f\in\mathcal{F}} \left[\pi_f(g,y) - \pi^N_f\right] \\
&= \prod_{f\in\mathcal{F}} \left\{ \left[\left(P^0 - B(G+y)\right)^T g_f - C_f(g_f)\right] - \pi^N_f \right\} \\
\text{subject to}\ & \pi_f(g,y) \geq (1-\delta)\tilde{\pi}^d_f(G_{-f}, y) + \delta\pi^N_f,\ \forall f\in\mathcal{F} \\
& g_f \in X_f,\ \forall f \in \mathcal{F} \\
& \text{ISO} \left\{ \begin{array}{ll} y \text{ free}, & P^0 - B(G+y) - \mu\mathbf{1} - \text{PTDF}^T\lambda = 0 \\ \mu \text{ free}, & \mathbf{1}^T y = 0 \\ \multicolumn{2}{l}{0 \leq \lambda \perp \text{PTDF}^T\lambda - T \leq 0} \end{array} \right\},
\end{aligned} \tag{17}$$

where the '$\perp$' sign means that the product of two vectors is 0. Note that some of the variables in (17) are redundant. Using similar algebraic derivation as in

Metzler et al. [31], we can obtain the following MPEC formulation of (17) with only the following variables: generation quantities g and the shadow price of transmission congestion λ.[11]

$$\begin{aligned} &\underset{g,\ \lambda}{\text{maximize}}\ \Theta(g,\lambda) = \prod_{f\in\mathcal{F}} [\pi_f(g,\lambda) - \pi_f^N] \\ &\text{subject to } (g,\lambda) \in \widetilde{\Omega}_\delta := \{g_f \in X_f,\ \forall f \in \mathcal{F} \\ &\qquad\qquad \pi_f(g,\lambda) \geq (1-\delta)\tilde{\pi}_f^d(G_{-f},\lambda) + \delta\pi_f^N,\ \forall f \in \mathcal{F} \\ &\qquad\qquad 0 \leq \lambda \perp r + \Delta G + M\lambda \geq 0\}\,, \end{aligned} \tag{18}$$

where

$$\pi_f(g,\lambda) = \kappa\mathbf{1}^T g_f - \omega G^T E g_f + \lambda^T \Delta g_f - C_f(g_f),\ \forall f \in \mathcal{F}, \tag{19}$$

and r, Δ, M, κ, ω, E are all input vectors or matrices with proper dimensions. The detailed derivation of (18) from (17) is provided in Appendix B, whereas the explicit formulation of $\tilde{\pi}_f^d(G_{-f},\lambda)$ is given in the next subsection.

Model (18) is a mathematical problem with linear complementarity constraints (MPLCP). The nonconvexity of the objective function and the complementarity constraints together make it difficult to find a globally optimal solution of the problem. The global optimization solver, BARON [39], is able to solve small instances of model (18). For larger instances, a branch-and-bound-based global optimization algorithm that exploits the model's special structure is developed in Liu [28]. Developing global optimization algorithms to solve the collusion model (18) is outside of the scope of this paper. We will use BARON to solve the numerical example to be presented in Section 5. Before that, however, some basic properties of the collusion model, including solution existence, are presented in the following subsection.

4.2 Model Properties

In this subsection we discuss properties associated with the collusion model (18), especially the non-emptiness of its feasible region and solution existence. First, we show that the matrix M in the complementarity constraint in (18) is positive semi-definite, which is useful for establishing other properties and for numerical implementation.

Lemma 1 *Given that $Q_i^0/P_i^0 > 0$ for all $i \in \mathcal{N}$, the matrix M in the following*

$$0 \leq \lambda \perp r + \Delta G + M\lambda \geq 0$$

as in (18) is symmetric positive semi-definite.

The proof is given in Appendix B as some notations needed for the proof are only provided there. We next discuss properties related to the optimal deviating payoff functions. In Section 3.2.2 we have discussed two formulations of the optimal value function: $\pi_f^d(G_{-f})$ in (15) and $\tilde{\pi}_f^d(G_{-f},a)$ in (16). There

[11] Note that the complementarity constraints in (18) are from the ISO's optimization problem, not from the smooth reformulation of the optimal value function of $\tilde{\pi}_f^d(G_{-f},a)$.

it is pointed out that due to the theoretical and computational difficulties associated with $\pi_f^d(G_{-f})$, we focus on the formulation $\tilde{\pi}_f^d(G_{-f}, a)$ (16). The vector reformulation of the collusion model can also be applied to $\tilde{\pi}_f^d(G_{-f}, a)$. With the parameter a replaced by λ, the resulting reformulation is as follows.

$$\begin{aligned}\tilde{\pi}_f^d(G_{-f}, \lambda) &= \max_{g_f^d \in X_f} \; \pi_f(g_f^d; G_{-f}, \lambda) \\ &= \max_{g_f^d \in X_f} \; \left[\kappa \mathbf{1}^T g_f^d - \omega(G_{-f} + g_f^d)^T E g_f^d + \lambda^T \Delta g_f^d - C_f(g_f^d)\right].\end{aligned} \tag{20}$$

We next show convexity and continuity of the implicit function $\tilde{\pi}_f^d(G_{-f}, \lambda)$. Convexity of the optimal deviation function facilitates computation, while continuity is needed to show the closedness of the feasible region in (18).

Lemma 2 *If the cost function $C_f(g_f)$ is strictly convex for each $f \in \mathcal{F}$, then $\tilde{\pi}_f^d(G_{-f}, \lambda)$ is continuous and convex with respect to (G_{-f}, λ).*

Proof. For a $f \in \mathcal{F}$, strict convexity of $C_f(g_f)$ implies that the firm's payoff function $\pi_f(\cdot; G_{-f}, \lambda)$ is strictly concave for each (G_{-f}, λ). In addition, $\pi_f(g_f; \cdot)$ is affine for all $g_f \in X_f$. Then the continuity and convexity of $\tilde{\pi}_f^d(G_{-f}, \lambda)$ follow directly from Proposition 1 in Harrington et al. [21]. $\square$

With the more compact, vector-based formulation of a deviating firm's optimization problem, we can revisit the relationship between the ideal formulation $\pi_f^d(G_{-f})$ and the relaxed formulation $\tilde{\pi}_f^d(G_{-f}, \lambda)$. Let $\Lambda_f^d(G_{-f})$ denote the feasible region of the optimization problem (15) that yields the optimal value function $\pi_f^d(G_{-f})$. By eliminating the redundant variables a_f^d, y_f^d, w_f^d, and p_f^d, the more compact form of the set $\Lambda_f^d(G_{-f})$ is as follows.

$$\Lambda_f^d(G_{-f}) = \{(g_f^d, \lambda_f^d) : g_f^d \in X_f, 0 \leq \lambda_f^d \perp r + (G_{-f} + g_f^d)^T \Delta g_f^d + M\lambda_f^d \geq 0\}. \tag{21}$$

Let $\bar{g}_f^{d^*}$ be an optimal solution to the simplified optimal deviating problem (20) with respect to a set of input parameter $(\bar{G}_{-f}, \bar{\lambda})$. If the cost function $C_f(\cdot)$ is strictly convex, the objective function in (20) is strictly concave. Consequently, the optimal solution $\bar{g}_f^{d^*}$ is unique. If $(\bar{g}_f^{d^*}, \bar{\lambda}) \in \Lambda_f^d(\bar{G}_{-f})$, then necessarily $\pi_f^d(\bar{G}_{-f}) \geq \tilde{\pi}_f^d(\bar{G}_{-f}, \bar{\lambda})$, which means that the unilateral one-time deviating profit is lower when the deviating firm does not consider the ISO's problem (and transmission constraints). Hence, naive deviating firms (not considering the ISO's problem) have less incentive to deviate than sophisticated deviating firms, and consequently, collusion would be easier to sustain with all deviating firms of the former type. This is not always the case, however. If $(\bar{g}_f^{d^*}, \bar{\lambda}) \notin \Lambda_f^d(G_{-f})$, the relationship between $\pi_f^d(\bar{G}_{-f})$ and $\tilde{\pi}_f^d(\bar{G}_{-f}, \bar{\lambda})$ is not clear. This is the compromise we need to make to achieve computation feasibility.

Now we focus on whether the feasible region $\widetilde{\Omega}_\delta$ in (18) is well-defined; namely, if the set is always nonempty for a $\delta \in [0, 1]$. Note that in building the collusion model (14), we discussed two single-stage Nash-Cournot models – one that leads to an EPEC model, and the other that results in a CP model, depending on the rationality assumptions of each power generation firm. If we use π_f^{EPEC} as the punishment-stage payoff, then if we can assume that an equilibrium exists to the static game (and let q^{EPEC} denote such an equilibrium), it is easy to see that $q^{EPEC} \in \Omega_\delta$ (not $\widetilde{\Omega}_\delta$), for any $\delta \in [0,1]$. This is essentially the result of Part (c) of Proposition 2 in Harrington et al. [21]. No definite results can be shown if $q^{EPEC} \in \widetilde{\Omega}_\delta$, however. If we use the CP model instead – namely, if we use π_f^{CP} as the punishment-stage payoff – then we can show that $\widetilde{\Omega}_\delta$ is always nonempty, as stated in the following proposition.

Proposition 2 *Given that an equilibrium exists to the static Nash-Cournot model with the exogenous-ISO assumption, then $\widetilde{\Omega}_\delta \neq \emptyset$ for each $\delta \in [0,1]$.*

Proof. Let $(g^{CP}, \lambda^{CP}, \gamma^{CP})$ denote an equilibrium to the exogenous-ISO model (with the formulation given in (23)), and let π_f^{CP} denote firm f's payoff in the corresponding equilibrium, for $f \in \mathcal{F}$. We show that $(g^{CP}, \lambda^{CP}) \in \widetilde{\Omega}_\delta$ for any $\delta \in [0,1]$. First, (g^{CP}, λ^{CP}) satisfies the complementarity constraint

$$0 \leq \lambda^{CP} \perp r + \Delta G^{CP} + M\lambda^{CP} \geq 0,$$

as shown in equation (15) in Metzler et al. [31]. Similarly, it is easy to see that given $(G_{-f}^{CP}, \lambda^{CP})$, g_f^{CP} satisfies the first-order optimality condition of Problem (20). Since (20) is a concave optimization problem with respect to a g_f^d, and its feasible region consists of linear constraints only, the first-order optimality condition is both necessary and sufficient for global optimality. As a result, we have that $\tilde{\pi}_f(G_{-f}^{CP}, \lambda^{CP}) = \pi_f(g_f^{CP}; G_{-f}^{CP}, \lambda^{CP}) = \pi_f^{CP}$. Then the incentive compatibility constraints are always binding at (g^{CP}, λ^{CP}) for any $\delta \in [0,1]$. Hence, $(g^{CP}, \lambda^{CP}) \in \widetilde{\Omega}_\delta$, $\forall \delta \in [0,1]$, and $\widetilde{\Omega}_\delta \neq \emptyset$ for all $\delta \in [0,1]$. $\square$

Though the feasible region of the collusion model is shown to be nonempty as long as we use the CP framework in the punishment stage, the proposition does not guarantee a finite optimal solution, due to the lack of explicit bounds on the variables λ.[12] Instead, we can solve the following convex optimization problem to find out if λ_k is bounded for each $k \in \mathcal{A}$.

$$\begin{aligned} \underset{g,\,\lambda,\,\varrho,\,\nu}{\text{maximize}}\ & \lambda_k \\ \text{subject to}\ & \lambda^T q + \sum_{l\in\mathcal{A}} \varrho_l + \lambda^T M \lambda \leq 0 \\ & \lambda \geq 0,\ r + \nu + M\lambda \geq 0 \\ & \nu = \Delta G \\ & g_f \in X_f,\ \forall f \in \mathcal{F},\ \lambda_l \underline{\nu}_l \leq \varrho_l \leq \lambda_l \overline{\nu}_l,\ \forall l \in \mathcal{A}, \end{aligned} \tag{22}$$

where ϱ_l's are auxiliary variables to replace the l-th element of the nonconvex term $\lambda^T \Delta G$, for each $l \in \mathcal{A}$, and $\underline{\nu}_l$, $\overline{\nu}_l$ represent the lower and upper bounds

[12] Though coerciveness of the objective function can guarantee a finite optimal solution without explicit boundedness, (see Proposition A.8 in Bertsekas [4]), the coerciveness of the Nash bargaining objective function cannot be easily shown.

of the other auxiliary variables ν_l, respectively. Since G is bounded both from above and below, $\underline{\nu}_l$ and $\overline{\nu}_l$ must be finite for all $l \in \mathcal{A}$. (22) is a convex optimization problem because the only nonlinear functions in the problem is the first set of constraints in (22), which are defined by convex quadratic functions as the matrix M is shown to be positive semi-definite in Lemma 1.

Below we show that if the objective function in (22) is bounded for each $k \in \mathcal{A}$, then so are λ_k's. Consequently, a finite optimal solution is obtainable for the collusion model (18) by the well-known Weierstrass' Extreme Value Theorem.

Proposition 3 *Assume that for each $k \in \mathcal{A}$, the corresponding optimization problem (22) is bounded. Let $\overline{\lambda}_k$ denote the value of the optimal solution of the k-th optimization problem. Then for any $(g, \lambda) \in \widetilde{\Omega}_\delta$, $\lambda_k \leq \overline{\lambda}_k$, $\forall k \in \mathcal{A}$.*

Proof. Let $(g^o, \lambda^o) \in \widetilde{\Omega}_\delta$. Define $\nu_k^o = (\Delta G)_k$, for each $k \in \mathcal{A}$. Further define that $\varrho_k^o = \lambda_k^o \nu_k^o$ for each $k \in \mathcal{A}$. Since $(g^o, \lambda^o) \in \widetilde{\Omega}_\delta$, $\lambda_k^o \geq 0$ for each k. Hence, $\lambda_k^o \underline{\nu}_k \leq \varrho_k^o \leq \lambda_k^o \overline{\nu}_k$, $\forall\ k \in \mathcal{A}$. Let $\mathcal{F}$ denote the feasible region of the convex problem (22). It is easy to see that $(g^o, \lambda^o, \varrho^o, \nu^o) \in \mathcal{F}$. Let $(g^*, \lambda^*, \varrho^*, \nu^*)$ denote an optimal solution to the convex problem (22), and e_k be a vector of a proper dimension whose k-th element is 1 and all other elements are 0. Then, by definition, we have that $\lambda_k^o = e_k^T \lambda^o \leq e_k^T \lambda^* = \overline{\lambda}_k$, $\forall k \in \mathcal{A}$. □

To better understand the set $\widetilde{\Omega}_\delta$, more properties are presented below.

Proposition 4 *Given that an equilibrium exists to the static Nash-Cournot model with the exogenous-ISO assumption, the following statements are true with $\delta \in [0, 1]$.*

(a) For all $(g, \lambda) \in \widetilde{\Omega}_\delta$, $\tilde{\pi}_f^d(G_{-f}, \lambda) \geq \pi_f(G, \lambda)$;
(b) For all $(g, \lambda) \in \widetilde{\Omega}_\delta$, $\tilde{\pi}_f^d(G_{-f}, \lambda) - \pi_f^{CP} \geq \pi_f(G, \lambda) - \pi_f^{CP} \geq 0$;
(c) For all $0 \leq \delta_1 \leq \delta_2 \leq 1$, $\widetilde{\Omega}_{\delta_1} \subset \widetilde{\Omega}_{\delta_2}$.

Proof. To prove part (a), notice that for all $(g, \lambda) \in \widetilde{\Omega}_\delta$, $g_f \in X_f$, the feasible region of the optimal deviation problem in (16). Then the inequality in (a) follows as $\tilde{\pi}_f^d(G_{-f}, \lambda)$ is the optimal value function of the function $\pi_f(g_f; G_{-f}, \lambda)$ over X_f. The proofs of (b) and (c) are exactly the same as those for part (a) and (c) in Proposition 2 of Harrington et al. [21]. □

The last property indicates that the region $\widetilde{\Omega}_\delta$ containing subgame perfect equilibria of a repeated game among power generators is expanding as the discount factor δ increases.

5 A Numerical Example

In this section we present a network example with two competing firms and five nodes. The network topology is shown in Figure 3. It is assumed that the reactances of the lines in the loop (1-2-3) are equal. Suppose that one firm has two generation units sitting at Node 1 and 2, while the other firm has one unit at Node 2. Assume that each firm's production cost function is a quadratic function; that is, $C_{fi}(g_{fi}) = MC_{fi}g_{fi} + \frac{1}{2}QC_{fi}g_{fi}^2$. Further let CAP$_{fi}$ denote firm f's generation capacity at node i. Each node has a demand represented by a linear inverse demand function (determined by two parameters P_i^0 and Q_i^0). The data related to the supply and demand are given in Table 1.

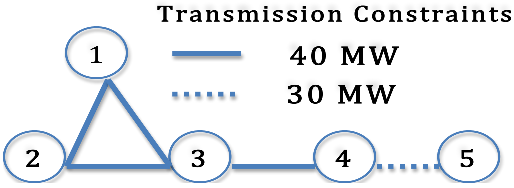

Fig. 3: An example – 5 nodes, 2 firms

Table 1: Input data of supply and demand for the numerical example

Node	Firm 1			Firm 2			Demand	
	MC [$/MWh]	QC [$/MWh^2]	CAP [MW]	MC [$/MWh]	QC [$/MWh^2]	CAP [MW]	P^0 [$/MWh]	Q^0 [MW]
1	15	0.02	150				40	250
2	15	0.02	50	18	0.01	100	35	200
3							32	320
4							30	300
5							40	200

Though an extremely simple example, the network in Figure 3 does contain two important features of transmission networks in the real world. First, it contains a loop, where flows within the loop need to satisfy Kirchhoff's Voltage Law. Such a property is a distinctive feature of electricity transmission networks, and it has been observed in real markets that strategic firms can exploit loop flow structures to enhance their ability to manipulate market prices (see, for example, Cicchetti et al. [10]). Second, there are two pure demand nodes at Node 4 and 5. Tight transmission constraints on the lines connecting the loads and generators would likely cause severe congestions on these lines. Power generators who are needed to serve these loads may therefore possess significant market power. Hence, the numerical results based on the network topology in Figure 3 can shed light on firms' behavior and market outcomes in a network with loop flows and severe transmission congestion.

To compare firms' equilibrium behavior under different levels of market competitiveness, we compare the results from perfect competition, static Nash-Cournot oligopoly and collusion. The results corresponding to generating firms' collusive behavior are computed based on the model in (18). As the model is nonconvex, we use the global optimization solver BARON.[13] The competitive

[13] The optimal value functions are written out explicitly through their KKT conditions. The complementarity constraints, with a generic form of $0 \leq f(x,y) \perp h(x,y) \geq 0$, are written as $f(x,y)^T h(x,y) \leq 0$. The resulting nonconvex optimization problem sent to BARON

market and the static Nash-Cournot equilibrium (based on the complementarity problem formulation in Metzler et al. [31]) are computed by the solver PATH [12]. The market outcomes are presented in Figure 4.[14] It is no surprise

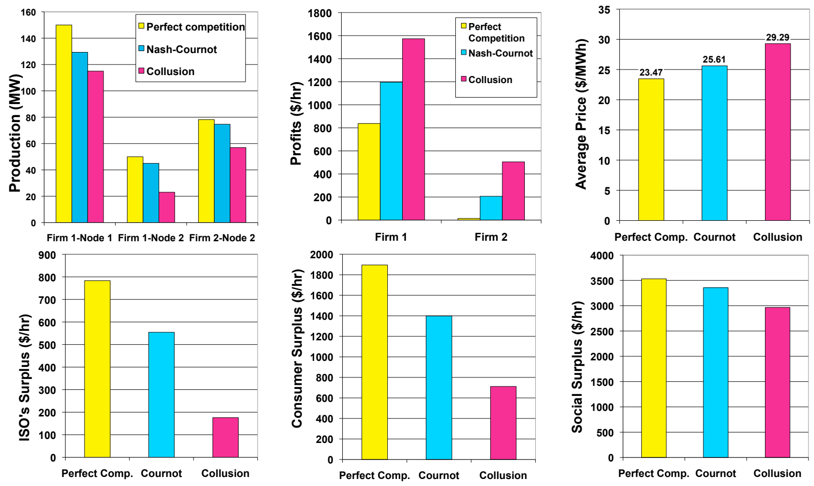

Fig. 4: Numerical results under different degrees of market competitiveness

to see that collusive firms can earn higher profits than in a static Nash-Cournot equilibrium by further reducing generation quantities. The interesting result is that the ISO's surplus when firms engage in a collusive game is much reduced compared to that in a Nash-Cournot equilibrium. To further investigate firms' behavior in the presence of transmission congestion, we show in Table 2 the net load flows on the transmission links and the corresponding shadow prices.

Table 2: Net flows on transmission lines and the corresponding shadow prices

	Arcs	Perfect Comp.	Nash-Cournot	Collusion ($\delta = .5$)
Net Flow	(1 2)	0 (0)	-2.59 (0)	-3.52 (0)
([MW])	**(2 3)**	**40 (12.86)**	**40 (10.12)**	**40 (0)**
	(3 1)	-40 (0)	-37.41 (0)	-36.48 (0)
	(3 4)	**40(1)**	**40 (0.74)**	**40 (0.65)**
(Shadow Price)	**(4 5)**	**30 (5)**	**30 (5)**	**30 (5)**
(λ, [$/MWh])	(2 1)	0 (0)	2.59 (0)	3.52 (0)
	(3 2)	-40 (0)	-40 (0)	-40 (0)
	(1 3)	**40 (1.97)**	**37.41 (0)**	**36.48 (0)**
	(4 3)	-40 (0)	-40 (0)	-40 (0)
	(5 4)	-30 (0)	-30 (0)	-30 (0)

Table 2 shows that within the loop of the transmission network (1-2-3), collusive generators can collectively act to decongest a line, hence diverting

(through NEOS server) consists of 40 variables and 67 constraints. The total solving time of this instance is 0.08 second, as BARON finds the optimal solution in preprocessing.

[14] The "Average Price" in Figure 4 and 5 equals $\sum_{i\in\mathcal{N}}[p_i \times (\sum_{f\in\mathcal{F}} g_{fi} + y_i)]/\sum_{i\in\mathcal{N}}\sum_{f\in\mathcal{F}} g_{fi}$.

part of the ISO's surplus into their own pockets. On the other hand, congestion on the lines connecting the load pockets (line 3-4 and 4-5) can only be slightly reduced (or remain the same). This is so because no generation can be used at nodes 4 or 5 to alter congestion. In other words, congestion on line 3-4 and line 4-5 is completely determined by the demand curves for nodes 4 and 5.

To gain more insight on the impacts of transmission constraints, we compare two versions of the previous example – one with transmission constraints and one without. Figure 5 shows the market outcomes of the two scenarios. For the scenario without transmission constraints (the darker bars in Figure

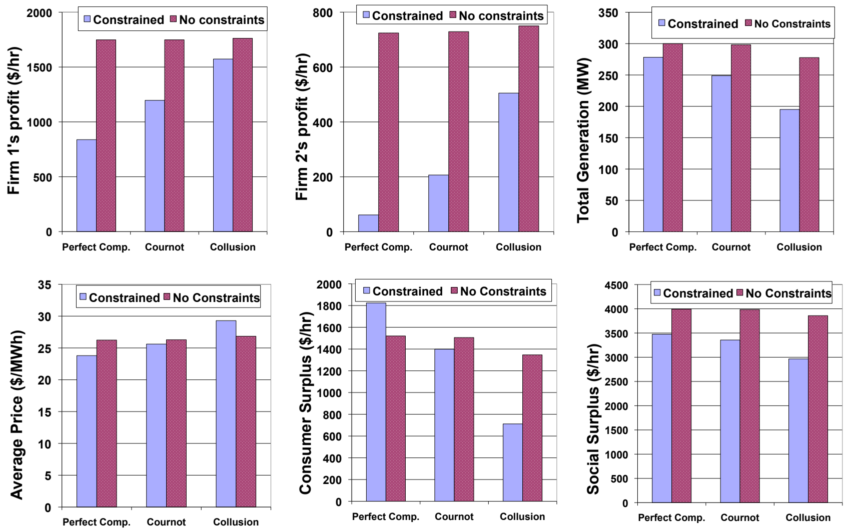

Fig. 5: Numerical comparison – with and without transmission constraints

5), firms' profits under imperfect competition, either Cournot or collusion, are limited due to their tight generation capacity constraints, together with relatively elastic demand. However, with tight transmission constraints (the lighter bars in Figure 5), firms' generation capacities become less important a constraint and hence, they have more room to manipulate their production quantities to exercise market power. Notice that under perfect competition, the average market price when there is transmission congestion is lower than the case without congestion.[15] This is so because with transmission constraints, generation capacities at Node 1 and 2 cannot all reach the other three demand nodes, which causes the prices to drop sharply at Node 1 and 2 due to the excess capacity. Prices certainly rise at the other three nodes. But the price drops at Node 1 and 2 outweigh the price increases at Nodes 3, 4 and 5, resulting in lower average price and higher consumer surplus, comparing

[15] The hub prices under the congested case are 28 \$/MWh, 28.26 \$/MWh and 28 \$/MWh, corresponding to a market of perfect competition, Nash-Cournot and collusion. Without congestion, the nodal prices are the same cross the network, and hence the hub price is the same as the average price reported in Figure 5.

to the case without transmission constraints. However, this relationship is reversed under collusion (that is, the average price is higher with transmission constraints) as collusive firms can strategically decongest line 1-3 and 2-3, together with strategically withholding production, causing sharp increases of prices at Node 1 and 2. This example illustrates when generation resources are tight, collusive firms' market power can be magnified by transmission constraints. To draw more definite policy conclusions, however, more numerical simulations are needed as one would expect that the location of power plants and loads, together with network topology, would also affect firms' strategies. Such analyses are deferred to future research.

6 Conclusion and Future Research

In this work, we have proposed an equilibrium-based model to simulate electric power generation firms' collusive behavior in a deregulated electricity market. The model is formulated as an optimization problem and is amenable to study heterogenous firms' collusive strategy through computation. In addition, the model explicitly incorporates transmission networks and other market players. It has been widely recognized that transmission constraints can be exploited by strategic firms to enhance their market power, and hence should to be incorporated in studies on generation firms' anti-competitive behavior. The numerical results presented in the paper do suggest that collusive generators can strategically exploit transmission congestion and reap additional profits compared to the situation without congestion. Hence, policy makers and market regulators need to pay special attention when designing market rules to lessen the possibility of collusion in a transmission-congested electricity market.

The tacit collusion model presented in this paper does suffer some analytical and practical difficulties. The inclusion of three distinct sets of market solutions – tacit collusion, unilateral profit maximizing, and single-stage noncooperative games – makes the model difficult to solve, while the lack of convexity of the optimization problem poses further computational challenges when globally optimal solutions are desired. Furthermore, the collusion model requires definition of additional parameters compared to static models: the length of the time lag before cheating is detected and rivals can react, and the relevant discount rate. These complications might limit the applicability of this modeling framework in practice, and in part explain why static Nash noncooperative models are much more popular in literature than dynamic models. Nonetheless, the current work represents the first attempt to establish an equilibrium-based dynamic model in electricity markets, and it can be easily enhanced with the advances in other related research areas such as EPECs and global optimization.

The current work can also be extended in several other ways. First, it is known from the seminal work by Green and Porter [19] that exogenous uncertainty (such as a demand shock) can induce price wars in a collusive game. It would be an interesting extension of the current model to include uncertainty and to study how that affects power generation firms' strategies.

Second, the model in this paper represents a repeated game; namely, the normal form of the static game being repeated remains the same throughout. A richer model may consider dynamic changes in the underlying static game, which leads to a dynamic game known as a supergame. One of such instances is a game with both spot and forward trading such that the forward contracts will limit the amount of spot market sales in future periods. Liu [29] has shown through computation that repeated (short-term) forward and spot market interaction may help sustain collusion, as the reversion to a static equilibrium in both markets once a deviation occurs is a more severe punishment than without the forward market. Such an effect might also occur for long-term forward contracts as they may lead to 'market splitting' behavior observed in other network-centric industries (airlines, telecommunication, etc) [2]. Another instance is that the number of firms may be changed with entry or exit decisions; firms' payoff functions (mainly the cost functions) may be affected by technology advancement; and firms' feasible action regions can be changed with capacity expansion decisions. Supergame models represent another significant extension of the current work and are subject to future research.

A final extension would be to consider tacit collusion in situations where transmission capacity and energy are sold in separate markets, rather than being combined as in ISO markets. Separate transmission and energy markets, together with imperfect arbitrage across those markets, open up the possibility of collusive geographical market sharing arrangements. One way to model this situation would be to modify the perfect arbitrage model of Liu [28] by either deleting the independent arbitragers and allowing generators themselves to sell in different markets (as in the spatial price discrimination model of Hobbs [23]), or assigning a cost to arbitrage.

Appendix A. Static Nash Equilibrium Models

Exogenous-ISO Model. For each power producer $f \in \mathcal{F}$, again let X_f denote its generic feasible production region. Then f solves the following optimization problem.

$$\underset{g_f \in X_f}{\text{maximize}}\ \pi_f(g_f) = \sum_{i \in \mathcal{N}} [P_i^0 - \frac{P_i^0}{Q_i^0}(\sum_{t \in \mathcal{F}} g_{ti} + y_i)] g_{fi} - \sum_{i \in \mathcal{N}} C_{fi}(g_{fi}) \tag{23}$$

The ISO solves the following optimization problem.

$$\begin{aligned} \underset{y}{\text{maximize}}\ & \sum_{i \in \mathcal{N}} \left[\int_0^{y_i + G_i} p_i(\tau_i) d\tau_i - \sum_{f \in \mathcal{F}} C_{fi}(g_{fi}) \right] \\ \text{subject to}\ & \sum_{i \in \mathcal{N}} y_i = 0 \\ & \sum_{i \in \mathcal{N}} \mathrm{PTDF}_{ki} y_i \leq T_k,\ \forall k \in \mathcal{A}. \end{aligned} \tag{24}$$

By writing out the the KKT conditions of each firm's and the ISO's optimization problem, together with the market clearing condition, we obtain a complementarity problem (CP).

Endogenous-ISO Model. For each power producer $f \in \mathcal{F}$, it solves the following optimization problem.

$$\begin{array}{l}\underset{g_f,\ y}{\text{maximize}}\ \pi_f(g_f, y) = \sum_{i\in\mathcal{N}}[P_i^0 - \frac{P_i^0}{Q_i^0}(\sum_{t\in\mathcal{F}} g_{ti} + y_i)]g_{fi} - \sum_{i\in\mathcal{N}} C_{fi}(g_{fi})\\ \text{subject to } g_f \in X_f,\\ \qquad\qquad \text{ISO}\left\{\begin{array}{l}\underset{y}{\text{maximize}}\ \sum_{i\in\mathcal{N}}\left[\int_0^{y_i+G_i} p_i(\tau_i)d\tau_i - \sum_{f\in\mathcal{F}} C_{fi}(g_{fi})\right]\\ \text{subject to } \sum_{i\in\mathcal{N}} y_i = 0\\ \qquad\qquad \sum_{i\in\mathcal{N}} \text{PTDF}_{ki} y_i \le T_k,\ \forall k \in \mathcal{A}\end{array}\right\}.\end{array} \tag{25}$$

As each firm's problem is an MPEC, by grouping all firms' problems together, we obtain an equilibrium problem with equilibrium constraints (EPEC).

Appendix B. Algebraic Reformulation of the Collusion Model

This appendix provides the derivation of the model (18) from (17). The key idea is that certain variables in the optimality conditions of the ISO's optimization problem are redundant. For the ease of argument, we re-provide the optimality conditions in the following.

$$y \text{ free},\quad P^0 - B(G+y) - \mu\mathbf{1} - \text{PTDF}^T\lambda = 0 \tag{26}$$

$$\mu \text{ free},\quad \mathbf{1}^T y = 0 \tag{27}$$

$$0 \le \lambda \perp \text{PTDF}^T\lambda - T \le 0, \tag{28}$$

which is derived under the affine inverse demand function assumption: $p(d) = P^0 - Bd$.

The purpose of the following derivation is to derive explicit expressions of y and μ with respect to G and λ, hence eliminating the need to keep the (redundant) variables y and μ in the model. Without loss of generality, we assume that node N is designated as the hub node, and introduce the following notations. Let $\breve{P}^0$, $\breve{G}$ and $\breve{y}$ denote the $\Re^{N-1}$ vectors excluding the N-th component of P^0, G and y, respectively. Also let P_N^0, Q_N^0 and G_N denote the N-th component of the vectors P^0, Q^0, and G, respectively. Further use $\breve{B}$ to denote the $(N-1)\times(N-1)$ matrix resulting from deleting the Nth row and column of B, and $\breve{\mathbf{1}}$ to represent an $(N-1)$ vector of 1's. Notice that by two equations in (26) and (27) we can have the following linear system of μ and $\breve{y}$ with G and λ as parameters:

$$\breve{P}^0 - \breve{B}(\breve{G}+\breve{y}) = \mu\breve{\mathbf{1}} + \text{PTDF}^T\lambda \tag{29}$$

$$P_N^0 - \frac{P_N^0}{Q_N^0}(G_N - \breve{\mathbf{1}}^T\breve{y}) = \mu. \tag{30}$$

where equation (29) is the same as in (26) for all the nodes other than the hub node; while equation (30) is derived using (26) at the hub node (where PTDF_{kN} for each $k \in \mathcal{A}$ is 0) and equation (27). Rewriting the linear system (29) and (30) into a matrix form yields the following.

$$\begin{bmatrix} \breve{B} & \breve{\mathbf{1}} \\ -\breve{\mathbf{1}}^T & \frac{Q_N^0}{P_N^0} \end{bmatrix} \begin{bmatrix} \breve{y} \\ \mu \end{bmatrix} = \begin{bmatrix} \breve{P}^0 - \breve{B}\breve{G} - \text{PTDF}^T\lambda \\ Q_N^0 - G_N \end{bmatrix}. \tag{31}$$

Recall that matrix $\breve{B} = \text{Diag}(b_i)$ is an $(N-1)\times(N-1)$ diagonal matrix, with $b_i = P_i^0/Q_i^0,\ i = 1,\ldots,N-1$. Under the assumption that $P_i^0 > 0$ and $Q_i^0 > 0$ for each $i = 1,\ldots,N$, $\breve{B}$ is positive definite, and so is the skew-symmetric coefficient matrix in (31) –

$$\begin{bmatrix} \breve{B} & \breve{\mathbf{1}} \\ -\breve{\mathbf{1}}^T & \frac{Q_N^0}{P_N^0} \end{bmatrix}.$$

Hence, the coefficient matrix is nonsingular, and the linear system (31) has a unique solution. Let $H \in \Re^{2L\times N}$ denote the PTDF matrix, and $\breve{H} \in \Re^{2L\times(N-1)}$ be the H matrix with its N-th column removed. Then the solution of (31) can be expressed as follows.

$$\mu = \kappa - \omega \mathbf{1}^T G - \breve{\rho}^T \breve{H}^T \lambda \tag{32}$$

$$\breve{y} = \breve{R}^0 + \breve{\Upsilon} G - \breve{\Xi} \breve{H}^T \lambda, \tag{33}$$

where

$$\omega = \frac{1}{\displaystyle\sum_{i=1}^{N} \frac{Q_i^0}{P_i^0}},\ \kappa = \omega \sum_{i=1}^{N} Q_i^0,$$

$$\breve{\rho} = \begin{bmatrix} \rho_1 \\ \vdots \\ \rho_{N-1} \end{bmatrix} \in \Re^{N-1} \text{ with } \rho_i = \omega \frac{Q_i^0}{P_i^0},\ \breve{R}^0 = \begin{bmatrix} R_1^0 \\ \vdots \\ R_{N-1}^0 \end{bmatrix} \in \Re^{N-1} \text{ with } R_i^0 = Q_i^0 - \frac{Q_i^0}{P_i^0}\kappa,$$

$$\breve{\Upsilon} \in \Re^{(N-1)\times N} \text{ with } \breve{\Upsilon}_{ij} = \begin{cases} \omega \dfrac{Q_i^0}{P_i^0} - 1,\ i = j \\ \omega \dfrac{Q_i^0}{P_i^0}, \quad i \neq j, \end{cases}$$

$$\breve{\Xi} \in \Re^{(N-1)\times(N-1)} \text{ with } \breve{\Xi}_{ij} = \begin{cases} \dfrac{Q_i^0}{P_i^0}(1-\rho_i),\ i = j \\ -\omega \dfrac{Q_i^0}{P_i^0}\dfrac{Q_j^0}{P_j^0},\ i \neq j. \end{cases}$$

With (32) and (33), the three conditions in the ISO's optimality conditions (26) – (28) can be condensed into one single complementarity constraint:

$$0 \leq \lambda \perp (T - H\breve{R}^0) - H\breve{\Upsilon}G + H\breve{\Xi}H^T\lambda \geq 0.$$

Let $r := T - H\breve{R}^0 \in \Re^{2L}$, $\Delta := -H\breve{\Upsilon} \in \Re^{2L\times N}$ and $M := H\breve{\Xi}H^T \in \Re^{2L\times 2L}$. The above complementarity system can then be written as

$$0 \leq \lambda \perp r + \Delta G + M\lambda \geq 0. \tag{34}$$

Furthermore, with (33) and equation (27), each firm's payoff function $\pi_f(g_f, y) = p(G + y)^T g_f - C_f(g_f)$ can be re-written as a function of (G, λ) as follows

$$\pi_f(g, \lambda) = \kappa \mathbf{1}^T g_f - \omega G^T E g_f + \lambda^T \Delta g_f - C_f(g_f),\ \forall f \in \mathcal{F}, \tag{35}$$

which is exactly equation (19). Hence, we have completed the derivation of (18) from (17).

Proof of Lemma 1. Given the assumptions in Lemma 1, the matrix $\breve{\Xi}$ is strongly diagonally dominant by the fact that for each $i = 1, \ldots, N-1$,

$$\Xi_{ii} = \frac{Q_i^0}{P_i^0}(1-\rho_i) = \frac{Q_i^0}{P_i^0}(1 - \omega\frac{Q_i^0}{P_i^0}) > \frac{Q_i^0}{P_i^0}(1 - \omega\frac{Q_i^0}{P_i^0} - \omega\frac{Q_N^0}{P_N^0}) = \sum_{i\neq j=1}^{N-1} |\Xi_{ij}|.$$

Since $\breve{\Xi}$ is also symmetric and has positive diagonal entries, it is then a positive definite matrix. Consequently, $M = H\breve{\Xi}H^T$ is a symmetric, positive semi-definite matrix. □

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