Math. Program., Ser. B 101: 57–94 (2004)

Digital Object Identifier (DOI) 10.1007/s10107-004-0537-4

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Spatial oligopolistic equilibria with arbitrage, shared resources, and price function conjectures*

Received: November 11, 2002 / Accepted: April 9, 2004
Published online: 7 July 2004 –

Abstract. This paper considers equilibria among multiple firms that are competing non-cooperatively against each other to sell electric power and buy resources needed to produce that power. Examples of such resources include fuels, power plant sites, and emissions allowances. The electric power market is a spatial market on a network in which flows are constrained by Kirchhoff's current and voltage laws. Arbitragers in the power market erase spatial price differences that are non-cost based. Power producers can compete in power markets à la Cournot (game in quantities), or in a generalization of the Cournot game (termed the conjectured supply function game) in which they anticipate that rivals will respond to price changes. In input markets, producers either compete à la Bertrand (price-taking behavior) or they can conjecture that price will increase with consumption of the resource. The simultaneous competition in power and input markets presents opportunities for strategic price behavior that cannot be analyzed using models of power markets alone. Depending on whether the producers treat the arbitrager endogenously or exogenously, we derive two mixed nonlinear complementarity formulations of the oligopolistic problem. We establish the existence and uniqueness of solutions as well as connections among the solutions to the model formulations. A numerical example is provided for illustrative purposes.

1. Introduction

Competition is being introduced into network-based industries throughout the globe with the objectives of lower costs and improving product quality and innovation [26]. Examples of such industries include telecommunications, transportation, and natural gas. As with other network-based industries, restructuring of the electric power generation sector has introduced market forces into an industry that was either subject to price controls and/or public ownership. In many power markets in Europe, North America, and South America, the result has been lowered prices for consumers [17].

But in a few places (notably California), supply shortages and inability of consumers to adjust their consumption to prices that vary greatly from hour to hour have

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Support was provided by the National Science Foundation under grant ECS-0080577.

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This author's research was partially supported by the National Science Foundation under grants ECS-0080577 and CCR-0098013.

* The authors thank Chris Day, Fieke Riekers, and Adrian Wals for their collaboration. They are particularly indebted to Grant Roch for writing AMPL and MATLAB codes for solving the numerical example reported in the paper.

resulted in large price increases. In California, a large share of the increases experienced in 2000–2001 has been blamed on market power: the ability of firms to cause prices to deviate from competitive levels by manipulating outputs or bids, or by other means [4, 19]. One way to exercise market power is to withhold output, for instance by declaring generators to be unavailable because of equipment failures. Prices can also be affected by strategic bidding, by manipulation of markets for needed inputs such as fuel or emissions allowances, and by deliberate congestion or decongestion of the network [3]. However, despite the infamous "Get Shorty", "Death Star", and other sophisticated strategies detailed in the infamous Enron memos, most of the market power problems experienced by California were due to simple economic and physical withholding of capacity [31]. Economic withholding occurs when generation capacity is made available only at high prices; physical withholding is when capacity unavailable at any price, for example by declaring an outage for maintenance at a time when demand is high. It has been argued that economic withholding in California was facilitated by contrived shortages of natural gas and NO_x allowances at crucial times [18, 20, 31].

Exercise of market power affects not only prices, profits, and consumer welfare, but can also decrease productive efficiency (if price increases encourage generation by high-cost smaller firms) while impacting the environment (if outputs shift among generators with different emission rates). Market power is universally viewed as one of the most serious imperfections in new power markets [14, 25, 30]. Therefore, models for projecting prices and other market outcomes should explicitly consider the potential for strategic behavior by power producers.

Many such models have been developed (see the reviews in [9, 11, 19]). Most are based on the calculation of Nash equilibria for a single time period for one commodity (electric energy). The most common Nash games simulated are those in which the firm's strategic variable is either sales and/or production quantities (Cournot games) or bid functions (supply function games). A new approach is the conjectured supply function [11](similar to the notion of conjectural variations, [13]). It can be viewed as a generalization of the Cournot model, in that generators can conjecture that rivals will respond in some *a priori* way to price changes rather than acting as if they assume that rivals will hold their quantities fixed. Approaches other than modeling that have been used to project market outcomes under conditions of market power include experiments with live subjects [24] and simulation models based on automata [5, 10].

Most models consider an undifferentiated energy market. However, in actual power markets, power (and its price) is differentiated by both time (e.g., hour of the day), space (location on the power network), and even "greenness" (renewable versus fossil-fuelled or nuclear energy). Furthermore, power markets are strongly linked to markets for other commodities, including ancillary services (such as operating reserves), fuels, and emissions allowances. Interactions among energy markets separated in space, time, or quality and between markets for energy and other commodities may present additional opportunities to exercise market power. These opportunities would be overlooked if markets were to be analyzed separately.

There have been some model-based analyses of these interactions. For instance, by selectively congesting or decongesting power lines (i.e., forcing flow constraints to be binding or slack), strategic power producers can isolate sub-markets from competition or deprive transmission owners of revenue they would normally receive [2, 7, 27]. Large hydropower producers can shift their generation from peak periods to offpeak periods in order to cause price spikes during the times of higher demand [6]. Strategic behavior in green and non-green energy markets has also been modeled [1]. Finally, as mentioned above, it has been argued that creation by power producers of artificial shortages in input markets (natural gas and NO_x emissions allowances) contributed to the high California prices of 2000–2001. Surprisingly, however, modelers have paid little attention to strategic manipulation of input and power markets (see [22] for an exception). Thus, there is a need for development, analysis, and application of such multimarket models.

In this paper, we propose models that represent the linkages between spatially separated markets for electricity and the resource inputs required for power production. These models explicitly recognize opportunities for simultaneous exercise of market power in more than one market. The models include representations of the following features of power markets:

- The predominance of bilateral trading of power between producers and consumers, as opposed to POOLCO-type arrangements, in which a single auctioneer buys all power from producers and then resells it to consumers;
- Strategic behavior in energy markets, represented by Cournot and conjectured supply function games [11];
- Demands by power producers for scarce transmission services that are allocated by an independent transmission system operator (ISO);
- Power transmission flows among spatial submarkets that are governed by Kirchhoff's voltage and current laws, modeled using a linearized DC network (as in [15]);
- Arbitrage by marketers among different power markets; and
- Competition by power producers and arbitragers for resource inputs, including various types of fuel and emissions allowances.

The models are introduced in Section 2, which describe each market participant's problem, including representations of the problems faced by the ISO, allocators of input resources, arbitragers, and power producers. Two versions of the power producer model are presented. In the first, producers anticipate how power will be redistributed by arbitragers among spatially separated markets as a result of changes in prices. This can be viewed as a Stackelberg game between producers (Stackelberg leaders) and arbitragers (Stackelberg followers). In the second version, producers instead view arbitrage as exogenous. These two types of games have previously been analyzed for the case of pure energy markets in [23]; this paper extend those results to consider how producers and arbitragers interact in input markets as well as energy markets. The market model is completed by imposing a set of consistency (market clearing) conditions on the participants.

The remainder of the paper analyzes certain properties of the models. Solution existence and uniqueness is first addressed for two variants of these models (Subsections 3.1 and 3.2). Generalizing the conjectured supply function models of [11], both variants assume energy producers anticipate that rivals will adjust their production and input consumption linearly if energy and input prices, respectively, change from their equilibrium levels. One variant assumes that the linear adjustment is defined by a function with a fixed price intercept and a slope that depends on the market price. The other variant, which is simpler to analyze, assumes instead that the slope is fixed and that the price intercept depends instead on the market price. Further results are obtained for the special case in which the arbitrager does not utilize resources, and producers use them only for production (Subsection 3.3). Section 4 provides a simple computational example that illustrates the model and the effect of resource constraints and expectations concerning resource prices on the solution. A set of conclusions closes the paper (Section 5).

2. The mathematical model

The spatial oligopolistic competition model considered herein is made up of four major components, each describing the behavior of an essential player in the market that is modeled by a linearized DC network [15]. Under the Nash equilibrium concept, these components are concatenated to yield a mixed linear or nonlinear complementarity formulation of the overall model.

There are four types of players: the producers, the ISO, the input resource allocator, and the arbitrager. The producers are firms that generate and sell a commodity (electric power); their primary decision variables are the sales at each point of consumption on the network, and generation by each of their power plants that are distributed on the network. These firms utilize market-allocated resources in their operations, the consumption of which is determined by the sales and generation. In addition, the firms need to pay the ISO for their use of the network to transfer power from generators to consumers. The market price of power at each location is determined by an affine demand function. In order to represent a firm's expectations concerning their rivals' reactions to prices, the model adopts a supply function conjecture that uses a first-order approximation near the equilibrium for the relationship between price and its rivals' sales in the region.

The producing firms are aware of the presence of an arbitrager in the market, who trades but does not produce power. We consider two alternative models: one in which the firms treat the arbitrager's actions endogenously and the other exogenously. In addition to the existence and uniqueness of solutions to the resulting models, we are also interested in the connection between their solutions.

2.1. The price function conjectures

We begin the mathematical formulation of the model with the introduction of two sets of price function conjectures: one for the price of power and the other for the resource price. We could also include a firm's conjecture concerning the price of transmission charged by the ISO; but since the treatment of this extension is similar, we do not develop the latter additional conjecture in the model.

The model postulates an affine demand function at each network node for power. In region $i = 1, \ldots, n$, the power price p_{fi} anticipated by producing firm $f \in \mathcal{F}$ is an affine function of the total sales S_i by all the firms plus the arbitrage amount a_{fi} anticipated by firm f, where $\mathcal{F}$ is a finite index set containing the labels of the firms. Specifically, we have

$$p_{fi} \equiv P_i^0 - \frac{P_i^0}{Q_i^0} (S_i + a_{fi}), \quad \forall\, i = 1, \ldots, n, \tag{1}$$

where P_i^0 and Q_i^0 are positive constants. By definition,

$$S_i \equiv \sum_{h \in \mathcal{F}} s_{hi}.$$

In the supply function conjecture model, the firms are assumed to anticipate that a deviation of the power price from its equilibrium level will stimulate a deviation in supply from rival firms from its equilibrium. In particular, the model postulates that the rival firms' sales

$$s_{-fi} \equiv \sum_{h \neq f, h \in \mathcal{F}} s_{hi}$$

are related to the price p_{fi} via the linear expression:

$$s_{-fi} = s^*_{-fi} + \beta_{fi}(p^*_i, s^*_{-fi})\,(\,p_{fi} - p^*_i\,), \tag{2}$$

where (p^*_i, s^*_{-fi}) is an equilibrium (price,sales) pair that is exogenous to firm f's profit maximization problem but is endogenous to the market, and the function $\beta_{fi}(x, y)$ is of one of two forms: (a) a positive constant β_{fi}, or (b) a rational function $y/(x - \alpha_{fi})$ for some positive constant α_{fi}. In case (a), we have (after rearrangement)

$$p_{fi} = \left(p^*_i - \frac{s^*_{-fi}}{\beta_{fi}} \right) + \frac{1}{\beta_{fi}}\, s_{-fi}. \tag{3}$$

With the term in the parenthesis on the right as exogenous to the firm but endogenous to the market, this expresses firm f's anticipated price p_{fi} as a linear function of its rivals' sales s_{-fi} with the fixed slope β_{fi}^{-1} and a variable intercept (variable relative to the market). In case (b), we have

$$p_{fi} = \alpha_{fi} + \frac{p^*_i - \alpha_{fi}}{s^*_{-fi}}\, s_{-fi}, \tag{4}$$

which expresses firm f's anticipated price p_{fi} as a linear function of its rivals' sales s_{-fi} with the fixed intercept α_{fi} and a variable slope (variable relative to the market, but exogenous to the producer). A noteworthy observation about the fixed-slope conjecture (3) is that two extreme specifications of β_{fi} yield two well-known models in the literature. Indeed, when $\beta_{fi} = 0$, we get $s_{-fi} = s^*_{-fi}$, which corresponds to the Cournot (fixed quantity) competition model. On the other hand, when $\beta_{fi} = \infty$, we get $p_{fi} = p^*_i$, which corresponds to the perfect competition model. By letting β_{fi} be a finite, positive constant, we obtain a range of models that vary between these two well-known cases.

In general, by substituting (2) into (1), we obtain

$$p_{fi} = P_i^0 - \frac{P_i^0}{Q_i^0} \left[s_{fi} + s^*_{-fi} + \beta_{fi}(p^*_i, s^*_{-fi})\,(\,p_{fi} - p^*_i\,) + a_{fi} \right]$$

which yields

$$p_{fi} = \frac{P_i^0 - \dfrac{P_i^0}{Q_i^0} \left[s_{fi} + s^*_{-fi} - \beta_{fi}(p_i^*, s^*_{-fi}) \, p_i^* + a_{fi} \right]}{1 + \dfrac{P_i^0}{Q_i^0} \beta_{fi}(p_i^*, s^*_{-fi})} \tag{5}$$

This is the effective demand function that firm f uses in its profit maximization problem. It shows the reaction of price to a firm's supply decision s_{fi}, accounting for both the response of consumers and the conjectured response of rival suppliers. Part of the goal of the analysis is to determine the range of values α_{fi} in a variable-slope supply function conjecture (4) that will ensure the well-definedness of the overall model. The following simple lemma is useful in the subsequent analysis.

Lemma 1. *Let*

$$\beta(x, y) \equiv \frac{y}{x - \alpha}, \qquad (x, y) \in \Re^2.$$

For every compact subset $U \subset \Re^2$, there exists a scalar $\bar{\alpha} > 0$ such that for every α such that $|\alpha| > \bar{\alpha}$, β is a well-defined Lipschitz continuous function on U with a Lipschitz constant $\tau(\alpha) > 0$ satisfying

$$\lim_{|\alpha| \to \infty} \tau(\alpha) = 0.$$

Proof. Since U is bounded, it is clear that for all α less than a threshold value, the denominator in $\beta(x, y)$ is nonzero for all (x, y) in U. Fix any such α. For any two pairs (x, y) and (x', y') in U, we have

$$\begin{aligned} \frac{y}{x - \alpha} - \frac{y'}{x' - \alpha} &= \frac{y - y'}{x - \alpha} - y' \left(\frac{1}{x' - \alpha} - \frac{1}{x - \alpha} \right) \\ &= \frac{y - y'}{x - \alpha} - y' \left(\frac{x - x'}{(x' - \alpha)(x - \alpha)} \right). \end{aligned}$$

Therefore,

$$\left| \frac{y}{x - \alpha} - \frac{y'}{x' - \alpha} \right| \leq [\, | y - y' | + | x - x' | \,] \left(\frac{1}{| x - \alpha |} + \frac{| y' |}{| x - \alpha | \, | x' - \alpha |} \right).$$

Since (x, y) and (x', y') are bounded, the existence of the desired $\tau(\alpha)$ follows readily. □

In addition to the supply function conjecture (1) in the energy commodity market, we also postulate that each firm has a conjecture regarding the response of the input resource price to changes in the quantity of resource the firm consumes. Specifically, for resource $j = 1, \ldots, m$, the response that firm f anticipates in the resource price ρ_{fj} in reaction to changes in firm f's consumption of resource j is given by the following first-order approximation around the equilibrium resource consumption:

$$\rho_{fj} = \rho_j^* + \sigma_{fj} \, (r_{fj} - r_{fj}^*),$$

where σ_{fj} is a nonnegative constant and the quantities with the asterisk indicate that they are variables exogenous to the generation firms but endogenous to the market. If $\sigma_{fj} = 0$, then this represents a Bertrand game, in which firms are price-takers with respect to the resource price. More generally, however, they anticipate that a resource's price increases if they consume more of it.

In essence, the above relationship is a fixed-slope, variable-intercept resource price function conjecture. We could also treat a variable-slope, fixed-intercept resource price function conjecture by an easy extension of the analysis that follows. Similarly, we could also include a functional conjecture on the transmission fee (as in [16]). For simplicity, all these variations and extensions are omitted.

2.2. *The ISO's problem*

Let $w \in \Re^n$ be defined as the vector of transmission fees charged by the ISO for use of the transmission network, where element w_i is the fee for transmitting 1 unit of energy from the network hub to node i. In a linearized DC network [29], the principle of superposition applies such that the cost of transporting power from i to i' equals $-w_i + w_{i'}$. As a result, the choice of hub is arbitrary. This unusual property (which, for instance, implies that the cost from i to i' is the negative of the cost from i' to i) differentiates electric power markets from spatial markets for other commodities.

The ISO is assumed to set the fees w^* in order to efficiently clear the market for transmission capacity. Alternatively, it might be assumed that there is a competitive market for transmission capacity in which transmission services are allocated to those firms that value them the most. Either assumption can be shown to be equivalent to modeling the ISO as a "price-taker" with respect to w^* [9]. Thus, taking w^* as exogenous to his problem, the ISO solves the following linear program to determine the energy commodity flows $y \in \Re^n$ in order to

$$\begin{aligned} &\text{maximize } y^T w^* \\ &\text{subject to } Hy \le h, \end{aligned}$$

where $H \in \Re^{\ell_0 \times n}$ is the technological matrix of the ISO and $h \in \Re^{\ell_0}$ is a given vector. In words, the ISO maximizes its revenue, subject to transmission constraints. Examples of such constraints in a power system could include thermal limits on flows in individual power lines, constraints on linear combinations of such flows (linearizations of so-called "nomograms"), and linear representations of flow control devices such as phase shifters. For simplicity, we have formulated the above problem using only inequality constraints; equality constraints do not affect the subsequent results. The optimality condition of the above linear program is: there exists a vector $z \in \Re^{\ell_0}$ such that

$$w^* = H^T z \quad \text{and} \quad 0 \le z \perp h - Hy \ge 0,$$

where the notation $z \perp w$ means that the vectors z and w are orthogonal.

2.3. *The resource allocator's problem*

The role of the resource allocator is analogous to that of the ISO in the following manner. The resource allocator can be viewed as a Walrasian auctioneer who sets the price of the resource to clear the market, or as a social planner who allocates the resource to those who value the resource the most. As an example, a government environmental agency might auction off a fixed amount (here designated as d) of emissions allowances that generators need to offset the pollutants they emit as result of the power production process. In the model, the resulting sales of allowances would be represented by the vector u. Alternatively a broker might be a pure facilitator of trades of, say, natural gas among buyers and sellers, in which case the amount of gas d he contributes to the market is zero. In that case, negative elements of u would represent sales by other parties, while positive elements would be purchases.

Proceeding in a manner similar to the ISO model development, we take the resource price $\rho^* \in \Re^m$ as exogenous to the resource allocator's problem. Therefore, the allocator solves the following linear program to determine the resource distributions $u \in \Re^m$ in order to

$$\begin{aligned} &\text{maximize } u^T \rho^* \\ &\text{subject to } \; Du \;\leq\; d, \end{aligned}$$

where $D \in \Re^{\ell_1 \times m}$ is the technological matrix of the resource allocator and $d \in \Re^{\ell_1}$ is the fixed vector of resources that the allocator can contribute to the market. These constraints can be viewed as market clearing conditions that ensure that the net amount of resources allocated to users Du do not exceed the resources available d. Like the ISO's problem, we have formulated the resource allocator's problem using only inequality constraints. The optimality condition of the latter linear program is: there exists a vector $v \in \Re^{\ell_1}$ such that

$$\rho^* \;=\; D^T v \quad \text{and} \quad 0 \leq v \perp d - Du \geq 0,$$

While the resource allocator's linear program may seem rather simplistic, it actually includes some useful special cases. For instance, in the realistic situation where the only constraint on the resources is that there is a fixed amount available for distribution, the constraint of the above resource problem becomes u not exceeding a given constant, which can easily be handled by our treatment.

A more general formulation that is a straightforward extension of the above model is the following. The allocator could have access to an elastic supply of the resource that could be used to meet the generators' demands; i.e., d could be a variable with a convex cost function. This could be used to simulate fuels markets, where higher fuel prices stimulate increased supply.

Of course, because of the linearity of the DC load flow model, this framework can also be used to model the ISO as an allocator of the transmission resource. However, we treat the transmission market separately because we wish to analyze the role of arbitrage among the spatially separated energy markets that transmission constraints create.

2.4. *The arbitrager's problem*

The arbitrager is assumed to be an entity that can buy power in one location and sell it in another. The only cost the arbitrager incurs is the ISO's transmission fees between the two locations and the cost of input resources required for that transaction. The arbitrager is assumed to be a price taker in all markets. As a result of this frictionless arbitrage, price differences in the network must reflect the cost of transmission (including perhaps the cost of input resources).

In the special case in which arbitrage requires no resources other than transmission capacity (as assumed in [16]), an implication of the linear DC transmission representation in the ISO's model is that such arbitrage will force the difference in price $p_{i'} - p_i$ between any two nodes i and i' of the network to precisely equal the cost of transmission from i to i' (i.e., $-w_i + w_{i'}$). Of course, this is not generally true of arbitrage among commodities; in the more general case, the price difference is constrained to be no more than the cost of transporting power from one location to the other, and could be less.

This efficient arbitrager is modeled as follows. Taking $p^* \in \Re^n$, $w^* \in \Re^n$, and $\rho^* \in \Re^m$ as exogenous to his problem, the arbitrager solves the following linear program to determine the arbitrage quantity $a \in \Re^n$ and resource usage $r^a \in \Re^m$ in order to

$$\begin{aligned}
&\text{maximize } a^T(\,p^* - w^*\,) - (\,r^a\,)^T \rho^* \\
&\text{subject to } Ga \;=\; Ge^0 \\
&\text{and} \qquad\quad E^a\, a \;=\; r^a + \omega^a,
\end{aligned}$$

where $\omega^a \in \Re^m$ is the pre-allocated resources owned by the arbitrager and $e^0 \in \Re^n$ is a given vector. The first constraint is a generalization of the condition that if an arbitrager sells energy, it must also buy an equal amount (on net, the arbitrager neither generates nor consumes energy). We assume throughout that the matrix $G \in \Re^{\ell_a \times n}$ has full row rank. The second constraint determines the amount of resources the arbitrager must buy as a function of the amount of energy arbitraged.

Substituting the variable $r^a = E^a a - \omega^a$ into the objective function, letting λ be the dual variable of the constraint $Ga = Ge^0$, and writing the optimality condition of the above simple linear program, we obtain:

$$\begin{bmatrix} 0 & -G^T \\ G & 0 \end{bmatrix} \begin{pmatrix} a \\ \lambda \end{pmatrix} = \begin{pmatrix} (\,E^a\,)^T \rho^* - p^* + w^* \\ Ge^0 \end{pmatrix}.$$

We should point out that unlike the ISO's problem and the resource allocator's problem, the technological constraint in the arbitrager's problem $Ga = Ge^0$ is an equality. In fact, inequality constraints in the latter problem would pose a technical difficulty when we consider the optimality conditions of the arbitrager's problem endogenous to the firms' problems, which we will describe next. Such inequality constraints present no difficulty in the exogenous-arbitrage version of the firms' problems; see Subsection 2.8.

2.5. Generation firm's problem: Endogenous arbitrage

Taking all quantities with an asterisk as exogenous to its problem, firm f solves a convex quadratic program to determine its sales $s^f \equiv (s_{fi}) \in \Re^n$, generation $g^f \equiv (g_{fi}) \in \Re^n$, resource usage $r^f \equiv (r_{fj}) \in \Re^m$, and anticipated arbitrage amount $a^f \equiv (a_{fi}) \in \Re^n$ in order to

$$\begin{aligned}
\text{maximize} \; & \sum_{i=1}^{n} [\, s_{fi}\, p_{fi} - c_{fi}(g_{fi}) - (\, s_{fi} - g_{fi}\,)\, w_i^*\,] \\
& \qquad - \sum_{j=1}^{m} [\, \rho_j^* + \sigma_{fj}(\, r_{fj} - r_{fj}^*\,)\,]\, r_{fj} \\
\text{subject to} \; & \sum_{i=1}^{n} s_{fi} \; = \; \sum_{i=1}^{n} g_{fi}, \\
& B^f s^f + E^f g^f \; = \; r^f + \omega^f, \\
& 0 \;\le\; g^f \;\le\; \text{CAP}^f, \\
& 0 \;\le\; s^f, \\
& \begin{bmatrix} 0 & -G^T \\ G & 0 \end{bmatrix} \begin{pmatrix} a^f \\ \lambda \end{pmatrix} = \begin{pmatrix} (\, E^a\,)^T \rho^* - p^f + w^* \\ G e^0 \end{pmatrix}, \\
\text{and} \qquad & p_{fi} \; = \; \frac{Q_i^0 - [\, s_{fi} + s_{-fi}^* - \beta_{fi}(p_i^*, s_{-fi}^*)\, p_i^* + a_{fi}\,]}{\dfrac{Q_i^0}{P_i^0} + \beta_{fi}(p_i^*, s_{-fi}^*)}, \\
& \qquad \forall\, i \;=\; 1, \ldots, n,
\end{aligned}$$

where $\omega^f \in \Re^m$ is the pre-allocated resources available to firm f, B^f and E^f are technological matrices, $\text{CAP}^f \in \Re^n$ is the vector of generation capacities, and $c_{fi} : \Re \to \Re$ is the generation cost function, which is assumed to be convex but not necessarily linear throughout the paper. This cost function excludes the cost of resources that are consumed during the production process. For instance, it is common to include fuel costs in $c_{fi}(g_{fi})$ (by multiplying the fuel use per unit by the fuel price); however, if fuel is modeled as a resource market, then it would not be so included.

The firm's objective can be interpreted as follows. The generator's profit depends on revenues from selling energy, the costs of generating and transmitting energy, and the net revenue from buying or selling in the input resources markets. For instance, input resources expenditures could include the cost of fuel or emissions allowances consumed in generation. As explained in Subsection 2.1, the price of resource j is a first-order approximation of how the equilibrium price ρ_j^* is conjectured by firm f to change if f changes the amount it consumes from the equilibrium value r_{fj}^*. The asterisk superscripts indicate that those variables are fixed parameters from the point of view of the firm f, even though they are variables from the point of view of the market.

Turning to the constraints, the first one ensures that sales are balanced by generation. The second defines the net amount of input resources r^f that the firm needs to buy from the resource allocator. Examples of such resources include fuel and emissions allowances. If fuel is an input, then the corresponding left-hand side denotes the fuel burnt during generation, with the appropriate entries in E^f then containing the fuel use per unit output. If emissions allowances are an input, then the corresponding entries in E^f contain instead the emissions rate per unit output. In both cases, the corresponding rows of B^f are zero. However, the latter rows could be nonzero, for instance if the input resource is a transmission right to the sales point (thus, transmission can be modeled as an input resource, rather than through the ISO's allocation system). In general, the distribution of pre-allocated resources ω^f (such as emissions allowances that are given annually by an environmental regulator to particular generating plants) can affect the solution because it can determine whether a firm is a net buyer or seller of power. This is because large net buyers will tend to act as oligopsonists (limiting their purchases to depress the price), while net sellers would instead be oligopolists (restricting supply in order to raise prices).

The third constraint limits energy generation from each plant to its capacity, while the fourth ensures that sales are nonnegative. The fifth set includes the first order conditions we defined in Subsection 2.4 for the arbitrager; that is, firm f anticipates how the arbitrager will react and reallocate power in response to price changes. Thus, this model can be a viewed as a special MPEC (Mathematical Program with Equilibrium Constraints) [21], in which f is a Stackelberg leader with respect to the arbitrager, who is a follower. Because the arbitrager's equilibrium conditions are linear equalities, firm f's MPEC is convex. The final set of constraints are the effective demand functions for power at each node i, (5). As we noted above, this relationship between the price of power and f's sales considers both the price elasticity of consumer demand and the conjectured supply response of rival power producers.

Variations in the above maximization problem can be handled by our treatment without too much difficulty. For instance, instead of the definitional equation on the resource usage:

$$B^f s^f + E^f g^f \;=\; r^f + \omega^f,$$

the model can easily accommodate an upper bounding constraint:

$$B^f s^f + E^f g^f \;\le\; r^f + \omega^f.$$

The only change is that an extra complementarity relation between a dual variable and the slack of the above inequality will appear in the final model formulation. For simplicity, we focus our treatment on the maximization problem as formulated above.

In what follows, we begin a series of algebraic derivations to simplify the formulation of the firms' maximization problems and thus the resulting model formulation. This exercise is carried out in order to facilitate the subsequent analysis of the overall model. However, from a computational point of view, the reductions obtained below do not necessarily provide the most suitable framework for solving the model. Part of the reason is that composite matrices are formed in the derivations that could easily destroy any data sparsity and/or special structure intrinsic to the model.

From the constraint defining p_{fi}, we obtain

$$a_{fi} \;=\; Q_i^0 - s_{fi} - s^*_{-fi} + \beta_{fi}(p_i^*, s^*_{-fi})\, p_i^* - \left(\frac{Q_i^0}{P_i^0} + \beta_{fi}(p_i^*, s^*_{-fi}) \right) p_{fi}. \quad (6)$$

Let

$$\tilde{\beta}_{fi}(x, y) \equiv \frac{Q_i^0}{P_i^0} + \beta_{fi}(x, y)$$

and let $\text{Diag}(\tilde{\beta}_{fi}(x, y))$ be the $n \times n$ diagonal matrix with diagonal entries $\tilde{\beta}_{fi}(x, y)$ for $i = 1, \ldots, n$. We write $\text{Diag}(\beta_{fi}(x, y))$ similarly. We have

$$\begin{aligned} Ge^0 &= G[\, Q^0 - s^f - s^{-f*} + \text{Diag}(\beta_{fi}(p_i^*, s_{-fi}^*))p^*\,] - G\text{Diag}(\tilde{\beta}_{fi}(p_i^*, s_{-fi}^*))p^f \\ &= -G[\, s^f + s^{-f*} - Q^0 - \text{Diag}(\beta_{fi}(p_i^*, s_{-fi}^*))p^*\,] \\ &\quad -G\text{Diag}(\tilde{\beta}_{fi}(p_i^*, s_{-fi}^*))[\,(E^a)^T\rho^* + w^* + G^T\lambda\,]. \end{aligned}$$

Assume for the moment that each quantity $\tilde{\beta}_{fi}(p_i^*, s_{-fi}^*)$ is positive for all $i = 1, \ldots, n$. This is clearly true if the function $\beta_{fi}(x, y)$ is a positive constant (as in the variable-intercept, fixed-slope conjecture). Subsequently, we will show that this is also true if $\beta_{fi}(x, y)$ is the rational function $y/(x - \alpha_{fi})$, provided that the constant α_{fi} is outside a certain finite range of values (see Subsection 3.1). Let

$$\boldsymbol{G}^f(p^*, s^{-f*}) \equiv G\text{Diag}(\tilde{\beta}_{fi}(p_i^*, s_{-fi}^*))G^T.$$

Since G has full row rank, the matrix $\boldsymbol{G}^f(p_i^*, s_{-fi}^*)$ is symmetric positive definite; it is therefore nonsingular. We deduce

$$\begin{aligned} \lambda = \boldsymbol{G}^f(p^*, s^{-f})^{-1} G\,[\, Q^0 - e^0 - (\, s^f + s^{-f*}\,) \\ +\text{Diag}(\beta_{fi}(p_i^*, s_{-fi}^*))p^* - \text{Diag}(\tilde{\beta}_{fi}(p_i^*, s_{-fi}^*))(\,(\,E^a\,)^T\rho^* + w^*\,)\,], \end{aligned}$$

which yields

$$\begin{aligned} p^f &= G^T\lambda + (\,E^a\,)^T\rho^* + w^* \\ &= G^T\boldsymbol{G}^f(p^*, s^{-f*})^{-1}G[\, Q^0 - e^0 - (\, s^f + s^{-f*}\,) + \text{Diag}(\beta_{fi}(p_i^*, s_{-fi}^*))p^*\,] \\ &\quad +[\, I - G^T\boldsymbol{G}^f(p^*, s^{-f*})^{-1}G\text{Diag}(\tilde{\beta}_{fi}(p_i^*, s_{-fi}^*))\,]\,[\,(\,E^a\,)^T\rho^* + w^*\,]. \end{aligned}$$

With p^f given by the above expression, firm f's problem can be written as

$$\begin{aligned} \text{maximize}\ & (\,s^f\,)^T p^f - \sum_{i=1}^n c_{fi}(g_{fi}) - (\,s^f - g^f\,)^T w^* - \\ & (B^f s^f + E^f g^f - \omega^f)^T[\rho^* + \text{Diag}(\sigma_{fj})(B^f s^f + E^f g^f - \omega^f - r^{f*})] \\ \text{subject to}\ & \sum_{i=1}^n s_{fi} = \sum_{i=1}^n g_{fi}, \quad (\varphi_f) \\ & g^f \le \text{CAP}^f, \quad (\eta^f) \\ & 0 \le s^f, g^f, \end{aligned}$$

where $r^{f*} \equiv (r_{fj}^*) \in \Re^m$ is the vector of firm f's equilibrium resource usage, and we write the dual variables in parentheses next to the corresponding constraints. The above problem is a concave maximization problem in the variables (s^f, g^f). Its Karush-Kuhn-Tucker conditions are:

$$
\begin{aligned}
&0 \le s^f \perp -p^f + A^f(p^*, s^{-f*})s^f + (B^f)^T \mathrm{Diag}(\sigma_{fj})(B^f s^f + E^f g^f - \omega^f) \\
&\quad +(B^f)^T[\rho^* + \mathrm{Diag}(\sigma_{fj})(B^f s^f + E^f g^f - \omega^f - r^{f*})] + w^* + \varphi_f \mathbf{1}_n \ge 0 \\
&0 \le g^f \perp \mathrm{vec}(c'_{fi}(g_{fi})) + (E^f)^T \mathrm{Diag}(\sigma_{fj})(B^f s^f + E^f g^f - \omega^f) \\
&\quad +(E^f)^T[\rho^* + \mathrm{Diag}(\sigma_{fj})(B^f s^f + E^f g^f - \omega^f - r^{f*})] \\
&\quad -w^* + \eta^f - \varphi_f \mathbf{1}_n \ge 0 \\
&0 \le \eta^f \perp \mathrm{CAP}^f - g^f \ge 0 \\
&0 = \sum_{i=1}^{n} (s_{fi} - g_{fi}),
\end{aligned}
$$

where

$$A^f(p^*, s^{-f*}) \equiv G^T \boldsymbol{G}^f(p^*, s^{-f*})^{-1} G$$

and $\mathrm{vec}(c'_{fi}(g_{fi}))$ is the n-vector whose i-th component is $c'_{fi}(g_{fi})$ for $i = 1, \ldots, n$ and $\mathbf{1}_n$ is the n-vector of all ones.

2.6. *Market clearing conditions*

.

At equilibrium, we postulate that the following conditions will hold:

- $p^f = p^*$, $r^f = r^{f*}$, and $\sum_{h \ne f, h \in \mathcal{F}} s^h = s^{-f} = s^{-f^*}$ for all $f \in \mathcal{F}$;
- $u = \sum_{f \in \mathcal{F}} r^f + r^a$;
- $y = \sum_{f \in \mathcal{F}} (s^f - g^f) + a$.

The first set of conditions imposes consistency on the expectations that each generating firm f has for power prices, its use of resources, and production by rival firms. That is, in equilibrium, the prices, resource use, and rival production that firm f anticipates (and treats as decision variables in its model) should be the same as their equilibrium values (and thus, in turn, the values anticipated by all other f). The second set of conditions clears the input resources market (i.e., what the market allocates equals that which is used by the producers and arbitrager). Associated with these market conditions is the vector of resource prices ρ^*; in equilibrium, they are set high enough to ensure that the quantity demanded of resources by producers equals the supply available. The final set does the same for the transmission market: the transmission service y that the ISO provides from the hub node to each node i equals the net amounts of power delivered by producers and arbitragers to that node. The transmission prices w^* are associated with this set of conditions. In equilibrium, they will be set so that demands for transmission service do not result in violation of the transmission constraints in the ISO's problem. If all of the ISO's transmission constraints are slack, those prices would be zero. But if under zero prices, one or more of the constraints would be violated, then w^* would be set at levels so that the transmission services demanded result in feasible flows.

Equating $p^f = p^*$, $s^{-f*} = s^{-f}$, letting

$$S \equiv \sum_{h \in \mathcal{F}} s^h,$$

and dropping the *, we obtain from (6),

$$a^f = Q^0 - S - \operatorname{Diag}(Q_i^0/P_i^0)\, p,$$

which shows that the anticipated arbitrage amount is the same for all firms. Repeating the above derivation but using the above simplified expression for a^f instead of (6), and letting

$$\boldsymbol{A} \equiv G^T (\, G \operatorname{Diag}(Q_i^0/P_i^0) G^T\,)^{-1} G \in \Re^{n \times n},$$

which is obviously symmetric positive semidefinite, we obtain

$$\begin{aligned} p &= \boldsymbol{A}\,(\, Q^0 - e^0 - S\,) + [\, I - \boldsymbol{A}\operatorname{Diag}(Q_i^0/P_i^0)\,]\,[\,(\, E^a\,)^T \rho^* + w^*\,] \\ &= \boldsymbol{A}\,(\, Q^0 - e^0 - S\,) + [\, I - \boldsymbol{A}\operatorname{Diag}(Q_i^0/P_i^0)\,]\,[\,(\, E^a\,)^T D^T v + H^T z\,], \end{aligned}$$

where we have used the optimality condition $\rho^* = D^T v$ in the resource allocator's problem and $w^* = H^T z$ in the ISO's problem. From the market clearing conditions, we obtain

$$\begin{aligned} u &= \sum_{f \in \mathcal{F}} r^f + r^a \\ &= \sum_{f \in \mathcal{F}} (\, B^f s^f + E^f g^f - \omega^f\,) + (\, E^a a - \omega^a\,) \\ &= \sum_{f \in \mathcal{F}} (\, B^f s^f + E^f g^f - \omega^f\,) + E^a (\, -S + Q^0 - \operatorname{Diag}(Q_i^0/P_i^0) p\,) - \omega^a \\ &= \tilde{q}_v + \sum_{f \in \mathcal{F}} (\, B^f s^f + E^f g^f\,) - E^a [\, I - \operatorname{Diag}(Q_i^0/P_i^0) \boldsymbol{A}\,] S \\ &\quad + E^a \operatorname{Diag}(Q_i^0/P_i^0) [\, \boldsymbol{A} \operatorname{Diag}(Q_i^0/P_i^0) - I\,]\,[\,(D E^a)^T v + H^T z\,] \end{aligned}$$

where

$$\tilde{q}_v \equiv E^a \left[\, Q^0 + \operatorname{Diag}(Q_i^0/P_i^0) \boldsymbol{A}\,(\, e^0 - Q^0\,) \right] - \sum_{f \in \mathcal{F}} \omega^f - \omega^a.$$

Furthermore,

$$
\begin{aligned}
y &= \sum_{f\in\mathcal{F}} (\, s^f - g^f \,) + a \\
&= Q^0 - \mathrm{Diag}(Q_i^0/P_i^0)\, p - \sum_{f\in\mathcal{F}} g^f \\
&= Q^0 + \mathrm{Diag}(Q_i^0/P_i^0)\boldsymbol{A}(\, e^0 - Q^0 + S\,) - \sum_{f\in\mathcal{F}} g^f \\
&\quad -\mathrm{Diag}(Q_i^0/P_i^0)[\, I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)\,]\,[\,(\, E^a\,)^T D^T v + H^T z\,],
\end{aligned}
$$

The above results allow elimination of y and u in the complementarity conditions for z and v in Subsections 2.2 and 2.3, respectively. Then, concatenating all the optimality conditions in the resource allocator's problem, the ISO's problem, and the firms' problems, we obtain the following mixed complementarity problem for the oligopolistic competition model with endogenous arbitrage:

$$
\begin{aligned}
& 0 \le v \;\perp\; q_v + DE^a\mathrm{Diag}(Q_i^0/P_i^0)[\, I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)\,]\,[\,(DE^a)^T v + H^T z\,] \\
& \quad - D\sum_{f\in\mathcal{F}} (\, B^f s^f + E^f g^f\,) + DE^a[\, I - \mathrm{Diag}(Q_i^0/P_i^0)\boldsymbol{A}\,]S \;\ge\; 0 \\
& 0 \le z \;\perp\; q_z + H\mathrm{Diag}(Q_i^0/P_i^0)[\, I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)\,]\,[\,(\, DE^a\,)^T v + H^T z\,] \\
& \quad - H\,\mathrm{Diag}(Q_i^0/P_i^0)\boldsymbol{A}S + H\sum_{f\in\mathcal{F}} g^f \;\ge\; 0
\end{aligned}
$$

and for every $f \in \mathcal{F}$:

$$
\begin{aligned}
& 0 \le s^f \;\perp\; q_s^f + \Big[\,(\, DB^f\,)^T - \{\, I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)\,\}(\, DE^a\,)^T\,\Big]\, v \\
& \quad + \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)H^T z + \boldsymbol{A}S \\
& \quad + [\, A^f(p, s^{-f}) + (\, B^f\,)^T\mathrm{Diag}(\sigma_{fj})B^f\,]s^f \\
& \quad + (\, B^f\,)^T\mathrm{Diag}(\sigma_{fj})E^f g^f + \varphi_f\,\mathbf{1}_n \;\ge\; 0 \\
& 0 \le g^f \;\perp\; q_g^f + (\, DE^f\,)^T v - H^T z + \mathrm{vec}(c'_{fi}(g_{fi})) \\
& \quad +(\, E^f\,)^T\mathrm{Diag}(\sigma_{fj})(\, B^f s^f + E^f g^f\,) + \eta^f - \varphi_f\,\mathbf{1}_n \;\ge\; 0 \\
& 0 \le \eta^f \;\perp\; \mathrm{CAP}^f - g^f \;\ge\; 0 \\
& 0 = \sum_{i=1}^{n} (\, s_{fi} - g_{fi}\,),
\end{aligned}
$$

where

$$
\begin{aligned}
q_s^f &\equiv \boldsymbol{A}(\, e^0 - Q^0\,) - (\, B^f\,)^T\mathrm{Diag}(\sigma_{fi})\omega^f \;\in\; \Re^n \\
q_g^f &\equiv -(\, E^f\,)^T\mathrm{Diag}(\sigma_{fi})\omega^f \;\in\; \Re^n \\
q_v &\equiv d - D\tilde{q}_v \;\in\; \Re^{\ell_1} \\
q_z &\equiv h - H[\, Q^0 + \mathrm{Diag}(Q_i^0/P_i^0)\boldsymbol{A}(\, e^0 - Q^0\,)\,] \;\in\; \Re^{\ell_0}.
\end{aligned}
$$

2.7. Endogenous arbitrage: compact formulation

To write the above formulation in a more compact form, we introduce some matrices. Let

$$\bar{\boldsymbol{M}} \equiv \begin{bmatrix} \boldsymbol{M}_v & \boldsymbol{M}_{vz} \\ \boldsymbol{M}_{zv} & \boldsymbol{M}_z \end{bmatrix} \equiv \begin{bmatrix} DE^a \\ H \end{bmatrix} \boldsymbol{B} \left[\, (DE^a)^T \;\; H^T \, \right] \in \Re^{(\ell_1+\ell_0)\times(\ell_1+\ell_0)},$$

where

$$\boldsymbol{B} \equiv \mathrm{Diag}(Q_i^0/P_i^0) - \mathrm{Diag}(Q_i^0/P_i^0)\boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0) \in \Re^{n\times n}. \tag{7}$$

Let

$$\begin{bmatrix} \boldsymbol{M}_s & \boldsymbol{M}_{sg} \\ \boldsymbol{M}_{gs} & \boldsymbol{M}_g \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{E}_{|\mathcal{F}|} \otimes \boldsymbol{A} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathrm{Diag}(M_s^f) & \mathrm{Diag}(M_{sg}^f) \\ \mathrm{Diag}(M_{gs}^f) & \mathrm{Diag}(M_g^f) \end{bmatrix} \in \Re^{2|\mathcal{F}|n\times 2|\mathcal{F}|n},$$

where $\boldsymbol{E}_{|\mathcal{F}|}$ is the square matrix of all ones of order $|\mathcal{F}|$, $\otimes$ denotes the Kronecker product,

$$\begin{bmatrix} M_s^f & M_{sg}^f \\ M_{gs}^f & M_g^f \end{bmatrix} \equiv \begin{bmatrix} (B^f)^T \\ (E^f)^T \end{bmatrix} \mathrm{Diag}(\sigma_{fi}) \left[\, B^f \;\; E^f \, \right] \in \Re^{2n\times 2n},$$

and $\mathrm{Diag}(M_s^f)$ is the block diagonal matrix with M_s^f as its diagonal blocks for $f \in \mathcal{F}$. Let

$$\begin{bmatrix} \boldsymbol{M}_{vs} & \boldsymbol{M}_{vg} \\ \boldsymbol{M}_{zs} & \boldsymbol{M}_{zg} \end{bmatrix} \equiv \begin{bmatrix} M_{vs}^1 & \cdots & M_{vs}^{|\mathcal{F}|} & M_{vg}^1 & \cdots & M_{vg}^{|\mathcal{F}|} \\ M_{zs} & \cdots & M_{zs} & H & \cdots & H \end{bmatrix} \in \Re^{(\ell_1+\ell_0)\times 2|\mathcal{F}|n)},$$

where for $f \in \mathcal{F}$,

$$\begin{aligned} M_{vs}^f &\equiv DE^a[\, I - \mathrm{Diag}(Q_i^0/P_i^0)\boldsymbol{A}\,] - DB^f \in \Re^{\ell_1\times n} \\ M_{vg}^f &\equiv -DE^f \in \Re^{\ell_1\times n} \\ M_{zs} &\equiv -H\mathrm{Diag}(Q_i^0/P_i^0)\boldsymbol{A} \in \Re^{\ell_0\times n}. \end{aligned}$$

Define the square matrix of order $(\ell_1 + \ell_2 + 3|\mathcal{F}|n + n)$:

$$\boldsymbol{M} \equiv \begin{bmatrix} \boldsymbol{M}_v & \boldsymbol{M}_{vz} & \boldsymbol{M}_{vs} & \boldsymbol{M}_{vg} & 0 & 0 \\ \boldsymbol{M}_{zv} & \boldsymbol{M}_z & \boldsymbol{M}_{zs} & \boldsymbol{M}_{zg} & 0 & 0 \\ -(\boldsymbol{M}_{vs})^T & -(\boldsymbol{M}_{zs})^T & \boldsymbol{M}_s & \boldsymbol{M}_{sg} & 0 & \boldsymbol{J} \\ -(\boldsymbol{M}_{vg})^T & -(\boldsymbol{M}_{zg})^T & \boldsymbol{M}_{gs} & \boldsymbol{M}_g & I_{|\mathcal{F}|n} & -\boldsymbol{J} \\ 0 & 0 & 0 & -I_{|\mathcal{F}|n} & 0 & 0 \\ 0 & 0 & -\boldsymbol{J}^T & \boldsymbol{J}^T & 0 & 0 \end{bmatrix},$$

where I_k is the identity matrix of order k and

$$\boldsymbol{J} \equiv \begin{bmatrix} \mathbf{1}_n & 0 & \cdots & 0 \\ 0 & \mathbf{1}_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}_n \end{bmatrix} \in \Re^{|\mathcal{F}|n \times |\mathcal{F}|};$$

let

$$\boldsymbol{q}_s \equiv \begin{pmatrix} q_s^1 \\ \vdots \\ q_s^{|\mathcal{F}|} \end{pmatrix} \quad \text{and} \quad \boldsymbol{q}_g \equiv \begin{pmatrix} q_g^1 \\ \vdots \\ q_g^{|\mathcal{F}|} \end{pmatrix}$$

Also define the nonlinear function $\boldsymbol{F}_s^{\mathrm{I}} : \Re^{(|\mathcal{F}|\times n)+\ell_1+\ell_0} \to \Re^{(|\mathcal{F}|\times n)}$ by

$$\boldsymbol{F}_s^{\mathrm{I}}(\boldsymbol{s}, v, z) \equiv \mathrm{Diag}(\, A^f(p, s^{-f})\,)\boldsymbol{s},$$

where $\boldsymbol{s}$ is the concatenated vector of the sales s^f for $f \in \mathcal{F}$,

$$A^f(p, s^{-f}) = G^T \left(G\mathrm{Diag}\left(\frac{Q_i^0}{P_i^0} + \beta_{fi}(p_i, s_{-fi}) \right) G^T \right)^{-1} G, \tag{8}$$

and $\mathrm{Diag}(A^f(p, s^{-f}))$ is the block diagonal matrix whose diagonal blocks are $A^f(p, s^{-f})$ for $f \in \mathcal{F}$. The dependence of $\boldsymbol{F}_s^{\mathrm{I}}$ on the pair (v, z) is through the price vector p, which is given by

$$p = \boldsymbol{A}\,(\,Q^0 - e^0 - S\,) + [\,I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)\,]\,[\,(\,DE^a\,)^T v + H^T z\,]. \tag{9}$$

With the above notations, we can state the compact, mixed NCP formulation of the oligopolistic competition problem with endogenous arbitrage:

$$\begin{array}{rcll} 0 & \leq v & \perp \\ 0 & \leq z & \perp \\ 0 & \leq \boldsymbol{s} & \perp \\ 0 & \leq \boldsymbol{g} & \perp \\ 0 & \leq \boldsymbol{\eta} & \perp \\ \text{free} & \varphi & \end{array} \begin{pmatrix} q_v \\ q_z \\ \boldsymbol{q}_s \\ \boldsymbol{q}_g \\ \mathbf{CAP} \\ 0 \end{pmatrix} + \boldsymbol{M} \begin{pmatrix} v \\ z \\ \boldsymbol{s} \\ \boldsymbol{g} \\ \boldsymbol{\eta} \\ \varphi \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{F}_s^{\mathrm{I}}(\boldsymbol{s}, v, z) \\ \mathrm{vec}(c'_{fi}(g_{fi})) \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \geq 0 \\ \geq 0 \\ \geq 0 \\ \geq 0 \\ \geq 0 \\ = 0. \end{array} \tag{10}$$

2.8. *Exogenous arbitrage*

In this model, each firm takes the arbitrage amounts a_i as exogenous to its profit maximization problem. This seemingly represents a less sophisticated power producer than in the endogenous arbitrage case, because in the latter situation each producing firm f anticipates how arbitragers will shift their purchases and sales in response to changes in f's sales. However, as we will show, there are close relationships between the solutions of the endogenous and exogenous arbitrage models. In particular, the market prices, firm profits, and firm production yielded by the two models are identical.

Specifically, taking a_i and all quantities with an asterisk as exogenous variables, firm f solves the following optimization problem for s^f, g^f, and r^f:

$$
\begin{aligned}
\text{maximize} \; & \sum_{i=1}^{n} [\, s_{fi}\, p_{fi} - c_{fi}(g_{fi}) - (\, s_{fi} - g_{fi}\,)\, w_i^*\,] \\
& \qquad\qquad - \sum_{j=1}^{m} [\, \rho_j^* + \sigma_{fj}(\, r_{fj} - r_{fj}^*\,)\,]\, r_{fj} \\
\text{subject to} \; & \sum_{i=1}^{n} s_{fi} \;=\; \sum_{i=1}^{n} g_{fi}, \\
& B^f s^f + E^f g^f \;=\; r^f + \omega^f, \\
& 0 \;\leq\; g^f \;\leq\; \text{CAP}^f, \\
& 0 \;\leq\; s^f, \\
\text{and} \qquad & p_{fi} \;=\; \frac{Q_i^0 - [\, s_{fi} + s_{-fi}^* - \beta_{fi}(p_i^*, s_{-fi}^*)\, p_i^* + a_i\,]}{\dfrac{Q_i^0}{P_i^0} + \beta_{fi}(p_i^*, s_{-fi}^*)}, \\
& \qquad \forall\, i \;=\; 1, \ldots, n,
\end{aligned}
$$

After eliminating the prices p_{fi} and resource usage r_{fj}, we obtain the Karush-Kuhn-Tucker conditions of the above concave maximization problem in the variable (s^f, g^f) as follow:

$$
\begin{aligned}
& 0 \leq s^f \;\perp\; -p^f + \text{Diag}(\tilde{\beta}_{fi}(p_i^*, s_{-fi}^*))^{-1} s^f \\
& \qquad + (\, B^f\,)^T \text{Diag}(\sigma_{fj})(\, B^f s^f + E^f g^f - \omega^f\,) \\
& \qquad + (\, B^f\,)^T [\, \rho^* + \text{Diag}(\sigma_{fj})(\, B^f s^f + E^f g^f - \omega^f - r^{f*}\,)\,] \\
& \qquad + w^* + \varphi_f\, \mathbf{1}_n \;\geq\; 0 \\
& 0 \leq g^f \;\perp\; \text{vec}(c'_{fi}(g_{fi})) + (\, E^f\,)^T \text{Diag}(\sigma_{fj})(\, B^f s^f + E^f g^f - \omega^f\,) \\
& \qquad + (\, E^f\,)^T [\, \rho^* + \text{Diag}(\sigma_{fj})(\, B^f s^f + E^f g^f - \omega^f - r^{f*}\,)\,] \\
& \qquad - w^* - \varphi_f\, \mathbf{1}_n \;\geq\; 0 \\
& 0 \leq \eta^f \;\perp\; \text{CAP}^f - g^f \;\geq\; 0 \\
& 0 = \sum_{i=1}^{n} (\, s_{fi} - g_{fi}\,).
\end{aligned}
$$

Invoking the market clearance conditions, we obtain

$$
a \;=\; Q^0 - S - \text{Diag}(Q_i^0 / P_i^0)\, p,
$$

$$
\begin{bmatrix} 0 & -G^T \\ G & 0 \end{bmatrix} \begin{pmatrix} a \\ \lambda \end{pmatrix} = \begin{pmatrix} (\, DE^a\,)^T v - p + H^T z \\ Ge^0 \end{pmatrix}.
$$

Solving these three equations for p, we obtain the same expression (9) for the price vector p. Setting $p^* = p^f = p$ in firm f's Karush-Kuhn-Tucker conditions and simplifying, we obtain the following mixed NCP formulation of the exogenous-arbitrage model:

$$\begin{aligned}
0 \le v \ \perp\ & q_v + DE^a \mathrm{Diag}(Q_i^0/P_i^0)[\, I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)\,]\,[\,(DE^a)^T v + H^T z\,] \\
& - D \sum_{h\in\mathcal{F}} (\, B^h s^h + E^h g^h \,) + DE^a[\, I - \mathrm{Diag}(Q_i^0/P_i^0)\boldsymbol{A}\,]S \ \ge\ 0 \\
0 \le z \ \perp\ & q_z + H\mathrm{Diag}(Q_i^0/P_i^0)[\, I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)\,]\,[\,(\, DE^a\,)^T v + H^T z\,] \\
& - H\,\mathrm{Diag}(Q_i^0/P_i^0)\boldsymbol{A}S + H \sum_{h\in\mathcal{F}} g^h \ \ge\ 0
\end{aligned}$$

and for every $f \in \mathcal{F}$:

$$\begin{aligned}
0 \le s^f \ \perp\ & q_s^f + \Big[\,(\, DB^f\,)^T - \{\, I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)\,\}(\, DE^a\,)^T\,\Big] v \\
& + \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)H^T z + \boldsymbol{A}S \\
& + [\,\mathrm{Diag}(\tilde{\beta}_{fi}(p_i, s_{-fi}))^{-1} + (\, B^f\,)^T \mathrm{Diag}(\sigma_{fj})B^f\,]s^f \\
& + (\, B^f\,)^T \mathrm{Diag}(\sigma_{fj})E^f g^f + \varphi_f\,\mathbf{1}_n \ \ge\ 0 \\
0 \le g^f \ \perp\ & q_g^f + (\, DE^f\,)^T v - H^T z + \mathrm{vec}(c'_{fi}(g_{fi})) \\
& + (\, E^f\,)^T \mathrm{Diag}(\sigma_{fj})(\, B^f s^f + E^f g^f\,) + \eta^f - \varphi_f\,\mathbf{1}_n \ \ge\ 0 \\
0 \le \eta^f \ \perp\ & \mathrm{CAP}^f - g^f \ \ge\ 0 \\
0 = & \sum_{i=1}^{n} (\, s_{fi} - g_{fi}\,).
\end{aligned}$$

In vector-matrix notation, the above problem is similar to (10), with just one important difference: namely, the function $\boldsymbol{F}_s^{\mathrm{I}}(\boldsymbol{s}, v, z)$ is replaced by

$$\boldsymbol{F}_s^{\mathrm{II}}(\boldsymbol{s}, v, z) \ \equiv\ \begin{pmatrix} \mathrm{Diag}(\tilde{\beta}_{1i}(p_i, s_{-1i}))^{-1}\, s^1 \\ \vdots \\ \mathrm{Diag}(\tilde{\beta}_{|\mathcal{F}|i}(p_i, s_{-|\mathcal{F}|i}))^{-1}\, s^{|\mathcal{F}|} \end{pmatrix}.$$

Specifically, the compact, mixed NCP formulation of exogenous-arbitrage model is as follows:

$$\begin{array}{rcll}
0 & \le & v & \perp \\
0 & \le & z & \perp \\
0 & \le & \boldsymbol{s} & \perp \\
0 & \le & \boldsymbol{g} & \perp \\
0 & \le & \boldsymbol{\eta} & \perp \\
\text{free} & & \varphi &
\end{array}
\begin{pmatrix} q_v \\ q_z \\ \boldsymbol{q}_s \\ \boldsymbol{q}_g \\ \mathbf{CAP} \\ 0 \end{pmatrix} + \boldsymbol{M} \begin{pmatrix} v \\ z \\ \boldsymbol{s} \\ \boldsymbol{g} \\ \boldsymbol{\eta} \\ \varphi \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{F}_s^{\mathrm{II}}(\boldsymbol{s}, v, z) \\ \mathrm{vec}(c'_{fi}(g_{fi})) \\ 0 \\ 0 \end{pmatrix}
\begin{array}{l} \ge\ 0 \\ \ge\ 0 \\ \ge\ 0 \\ \ge\ 0 \\ \ge\ 0 \\ =\ 0. \end{array} \qquad (11)$$

3. Solution existence and uniqueness

There are altogether four models described by the two NCPs (10) and (11), corresponding to two forms of the function $\beta_{fi}(x, y)$: a constant or a simple rational function. With the two NCPs as the basic formulations for the models, the existence and uniqueness of a solution to the models therefore hinge critically on the properties of the matrix $\boldsymbol{M}$ and the functions $\boldsymbol{F}_s^{\mathrm{I,II}}(\boldsymbol{s}, v, z)$. As a first step in this analysis, we show that the matrix $\boldsymbol{B}$ given by (7) is positive semidefinite.

Proposition 1. Assume that G has full row rank. The matrix

$$\boldsymbol{B} \equiv \mathrm{Diag}(Q_i^0/P_i^0) - \mathrm{Diag}(Q_i^0/P_i^0)\boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)$$

is symmetric positive semidefinite. Moreover,

$$\boldsymbol{B}x = 0 \Leftrightarrow x = G^T y \text{ for some } y;$$

finally, there exists a constant $\delta > 0$ such that

$$x^T \boldsymbol{B} x \geq \delta \, \| \, [\, I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)\,]x \, \|^2.$$

Proof. The displayed matrix is the Schur complement of $G\mathrm{Diag}(Q_i^0/P_i^0)G^T$ in

$$\begin{bmatrix} G\mathrm{Diag}(Q_i^0/P_i^0)G^T & G\mathrm{Diag}(Q_i^0/P_i^0) \\ \mathrm{Diag}(Q_i^0/P_i^0)G^T & \mathrm{Diag}(Q_i^0/P_i^0) \end{bmatrix} = \begin{bmatrix} G \\ I \end{bmatrix} \mathrm{Diag}(Q_i^0/P_i^0) \begin{bmatrix} G^T & I \end{bmatrix}.$$

Since the latter matrix is clearly positive semidefinite, it follows readily that $\boldsymbol{B}$ is positive semidefinite. To prove the second assertion, suppose $\boldsymbol{B}x = 0$. Let

$$y \equiv (G\mathrm{Diag}(Q_i^0/P_i^0)G^T)^{-1} G\mathrm{Diag}(Q_i^0/P_i^0)x.$$

It follows that

$$\begin{bmatrix} G\mathrm{Diag}(Q_i^0/P_i^0)G^T & G\mathrm{Diag}(Q_i^0/P_i^0) \\ \mathrm{Diag}(Q_i^0/P_i^0)G^T & \mathrm{Diag}(Q_i^0/P_i^0) \end{bmatrix} \begin{pmatrix} -y \\ x \end{pmatrix} = 0.$$

Hence $G^T y = x$. Conversely, if $x = G^T y$, then it is easy to show that $\boldsymbol{B}x = 0$.

To prove the third assertion, write $\Lambda \equiv \mathrm{Diag}(Q_i^0/P_i^0)$. With $y \equiv \Lambda x$, we have

$$\begin{aligned} x^T \boldsymbol{B} x &= y^T [\, \Lambda^{-1} - \boldsymbol{A} \,] y \\ &\geq \eta \, \left\| \, [\, \Lambda^{-1} - \boldsymbol{A} \,] y \, \right\|^2 = \eta \, \left\| \, [\, I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)\,]x \, \right\|^2, \end{aligned}$$

where η is the reciprocal of the largest eigenvalue of the symmetric positive semidefinite matrix $\Lambda^{-1} - \boldsymbol{A}$. □

It follows easily from the above proposition that the matrix $\boldsymbol{M}$ is positive semidefinite (as proved below), albeit not symmetric

Proposition 2. If G has full row rank, then $\boldsymbol{M}$ is positive semidefinite.

Proof. Clearly the matrix $\bar{\boldsymbol{M}}$ is symmetric positive semidefinite. Since $\boldsymbol{E}_{|\mathcal{F}|}$ and $\boldsymbol{A}$ are symmetric positive semidefinite, so is their Kronecker product. Moreover, since

$$\begin{bmatrix} M_s^f & M_{sg}^f \\ M_{gs}^f & M_g^f \end{bmatrix}$$

is symmetric positive semidefinite, so is

$$\begin{bmatrix} \mathrm{Diag}(M_s^f) & \mathrm{Diag}(M_{sg}^f) \\ \mathrm{Diag}(M_{gs}^f) & \mathrm{Diag}(M_g^f) \end{bmatrix}.$$

Hence the matrix

$$\begin{bmatrix} \boldsymbol{M}_s & \boldsymbol{M}_{sg} \\ \boldsymbol{M}_{gs} & \boldsymbol{M}_g \end{bmatrix},$$

is symmetric positive semidefinite. The positive semidefiniteness of $\boldsymbol{M}$ follows readily. □

As to the functions $\boldsymbol{F}_s^{\mathrm{I,II}}$, the form of the price conjecture function $\beta_{fi}(x, y)$ plays an essential role. If the latter function is a positive constant (the fixed-slope, variable-intercept conjecture (3)), then clearly $\boldsymbol{F}_s^{\mathrm{I,II}}(\boldsymbol{s}, v, z)$ are linear functions in $\boldsymbol{s}$ only. Moreover, in this case $\boldsymbol{F}_s^{\mathrm{II}}(\boldsymbol{s}, v, z)$ is a strongly monotone linear function of $\boldsymbol{s}$, while $\boldsymbol{F}_s^{\mathrm{I}}(\boldsymbol{s}, v, z)$ is a monotone linear function of $\boldsymbol{s}$. Thus, the complementarity problems resulting from the fixed-slope, variable-intercept price function conjecture (3) is much simpler than those from the variable-slope, fixed-intercept conjecture (4). In the next subsection, we treat the latter conjecture and postpone the analysis of the former conjecture until Subsection 3.2. It should be noted that since we allow the generation cost function $c_{fi}(g_{fi})$ to be nonlinear, the resulting formulations (10) and (11) remain NCPs under either price function conjecture.

3.1. The variable-slope, fixed-intercept conjecture

In this subsection, we show that even under the variable-slope, fixed-intercept conjecture where

$$\beta_{fi}(x, y) \equiv \frac{y}{x - \alpha_{fi}}, \qquad (x, y) \in \Re^2,$$

there exists a range of values of the constant α_{fi} outside which $\boldsymbol{F}_s^{\mathrm{I,II}}(\boldsymbol{s}, v, z)$ are well-defined functions of the pair $(\boldsymbol{s}, \boldsymbol{g})$ in a certain domain and with (v, z) substituted by a certain implicit function of $(\boldsymbol{s}, \boldsymbol{g})$. To formally state and establish this claim, we begin by quoting a result from the theory of monotone linear complementarity problems (LCPs). Part (a) of this result is well-known (see, e.g., [8]); part (b) is less well-known, but its proof can be found in [12]. We now explain some notation. For an arbitrary $N \times N$

matrix $\boldsymbol{M}$, let $\mathcal{R}(\boldsymbol{M})$ be the LCP range of $\boldsymbol{M}$; that is, $\boldsymbol{q} \in \mathcal{R}(\boldsymbol{M})$ if and only if the LCP $(\boldsymbol{q}, \boldsymbol{M})$

$$0 \leq z \perp \boldsymbol{q} + \boldsymbol{M}z \geq 0$$

has a solution. When $\boldsymbol{M}$ is symmetric positive semidefinite (as in our application below), it can be shown that

$$\mathcal{R}(\boldsymbol{M}) = \{ \boldsymbol{q} \in \Re^N : \exists z \in \Re^N \text{ satisfying } \boldsymbol{q} + \boldsymbol{M}z \geq 0 \}.$$

Proposition 3. Let $\boldsymbol{M} \equiv A^T \Xi A$, where Ξ is a symmetric positive semidefinite $m \times m$ matrix and A is an arbitrary $m \times n$ matrix.

(a) For every $\boldsymbol{q} \in \mathcal{R}(\boldsymbol{M})$, if z^1 and z^2 are any two solutions of the LCP $(\boldsymbol{q}, \boldsymbol{M})$, then $\Xi A z^1 = \Xi A z^2$. Let $\tilde{w}(\boldsymbol{q})$ denote the common vector $\Xi A z$ for any solution z of the LCP $(\boldsymbol{q}, \boldsymbol{M})$.
(b) The function $\tilde{w} : \mathcal{R}(\boldsymbol{M}) \to \Re^n$ is Lipschitz continuous on its domain; that is, there exists a constant $\tau > 0$ such that for every $\boldsymbol{q}^1$ and $\boldsymbol{q}^2$ in $\mathcal{R}(\boldsymbol{M})$,

$$\| \tilde{w}(\boldsymbol{q}^1) - \tilde{w}(\boldsymbol{q}^2) \| \leq \tau \| \boldsymbol{q}^1 - \boldsymbol{q}^2 \|.$$

In particular, it holds that

$$\| \tilde{w}(\boldsymbol{q}) \| \leq \tau \| \boldsymbol{q} \|, \quad \forall \boldsymbol{q} \in \mathcal{R}(\boldsymbol{M}).$$

We apply this proposition to the following parametric LCP in the primary variable (v, z) and with $(\boldsymbol{s}, \boldsymbol{g})$ as the parameter:

$$\begin{array}{c} 0 \leq \\ 0 \leq \end{array} \begin{pmatrix} v \\ z \end{pmatrix} \begin{array}{c} \perp \\ \perp \end{array} \begin{pmatrix} q_v \\ q_z \end{pmatrix} + \begin{bmatrix} \boldsymbol{M}_v & \boldsymbol{M}_{vz} \\ \boldsymbol{M}_{zv} & \boldsymbol{M}_z \end{bmatrix} \begin{pmatrix} v \\ z \end{pmatrix} + \begin{bmatrix} \boldsymbol{M}_{vs} & \boldsymbol{M}_{vg} \\ \boldsymbol{M}_{zs} & \boldsymbol{M}_{zg} \end{bmatrix} \begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{g} \end{pmatrix} \begin{array}{c} \geq 0 \\ \geq 0. \end{array} \tag{12}$$

Recalling the notation $\bar{\boldsymbol{M}}$ which denotes the defining matrix of the above parametric LCP, we let

$$\Omega \equiv \left\{ (\boldsymbol{s}, \boldsymbol{g}) \in \Re_+^{2|\mathcal{F}|n} : g^f \leq \text{CAP}^f, \quad \sum_{i=1}^n (s_{fi} - g_{fi}) = 0, \quad \forall f \in \mathcal{F}; \right.$$
$$\left. \text{and } \begin{pmatrix} q_v \\ q_z \end{pmatrix} + \begin{bmatrix} \boldsymbol{M}_{vs} & \boldsymbol{M}_{vg} \\ \boldsymbol{M}_{zs} & \boldsymbol{M}_{zg} \end{bmatrix} \begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{g} \end{pmatrix} \in \mathcal{R}(\bar{\boldsymbol{M}}) \right\}.$$

For the time being, we assume that the set Ω is nonempty. (Clearly, this is a necessary condition for the models to have a solution. See Subsection 3.4 for a treatment of this key nonemptiness issue.) Since $\mathcal{R}(\bar{\boldsymbol{M}})$ is a polyhedron, Ω is a polytope. By Proposition 3, there exists a constant $\tau > 0$ such that for every $(\boldsymbol{s}, \boldsymbol{g}) \in \Omega$ and for any two solutions (v^1, z^1) and (v^2, z^2) of the LCP (12), we have

$$\boldsymbol{B}[(DE^a)^T v^1 + H^T z^1] = \boldsymbol{B}[(DE^a)^T v^2 + H^T z^2];$$

moreover, for any $(\boldsymbol{s}', \boldsymbol{g}') \in \Omega$,

$$\left\| \boldsymbol{B} \left[(DE^a)^T (v^1 - v') + H^T (z^1 - z') \right] \right\| \leq \tau [\, \| \boldsymbol{s} - \boldsymbol{s}' \| + \| \boldsymbol{g} - \boldsymbol{g}' \| \,],$$

where (v', z') is any solution of the LCP (12) corresponding to $(\boldsymbol{s}', \boldsymbol{g}')$. Recalling the definition of the matrix $\boldsymbol{B}$ in (7), we deduce

$$\begin{aligned}&\| \left[I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0) \right] \left[(DE^a)^T (v^1 - v') + H^T (z^1 - z') \right] \| \\ &\quad \le \tau' [\, \| \boldsymbol{s} - \boldsymbol{s}' \| + \| \boldsymbol{g} - \boldsymbol{g}' \| \,],\end{aligned}$$

where $\tau' > 0$ is a constant that depends only on the model data. From (9) we recall that

$$p \;=\; \boldsymbol{A}\,(Q^0 - e^0 - S) + [\, I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0) \,]\,[\, (DE^a)^T v + H^T z \,];$$

consequently, the function $p : (\boldsymbol{s}, \boldsymbol{g}) \in \Omega \to \Re^n$, with $p(\boldsymbol{s}, \boldsymbol{g})$ given by the above expression for any solution (v, z) of the LCP (12) corresponding to $(\boldsymbol{s}, \boldsymbol{g})$, is Lipschitz continuous on its domain. Since Ω is compact, it follows that there exists a constant $\bar{\alpha}$ such that for all $|\alpha_{fi}| > \bar{\alpha}$, the function

$$(\boldsymbol{s}, \boldsymbol{g}) \;\mapsto\; \tilde{\beta}_{fi}(p_i, s_{-fi}) \;=\; \frac{Q_i^0}{P_i^0} + \frac{s_{-fi}}{p_i(\boldsymbol{s}, \boldsymbol{g}) - \alpha_{fi}}$$

is positive on Ω. Thus the matrix $\mathrm{Diag}(\tilde{\beta}_{fi}(p_i, s_{-fi}))$ and also its inverse are both positive definite; hence the matrix $A^f(p, s^{-f})$ given by (8) is well defined and positive semidefinite for all f. Consequently, the two nonlinear functions $\boldsymbol{F}_s^{\mathrm{I,II}}(\boldsymbol{s}, v, z)$ are well defined and can be expressed as an implicit, Lipschitz continuous function of $(\boldsymbol{s}, \boldsymbol{g})$ on the domain Ω.

Based on the above preliminary analysis, we can establish the following existence result for the variable-slope, fixed-intercept models.

Theorem 1. Assume that G has full row rank and that each cost function $c_{fi}(g_{fi})$ is continuously differentiable. Suppose that the set Ω is nonempty and that $|\alpha_{fi}| > \bar{\alpha}$ for all $f \in \mathcal{F}$ and all $i = 1, \cdots, n$. With the variable-slope, fixed-intercept conjecture (4), both the exogenous-arbitrage model and the endogenous-arbitrage model have equilibrium solutions.

Proof. In view of above analysis, we write $\boldsymbol{F}_s^{\mathrm{I,II}}(\boldsymbol{s}, \boldsymbol{g})$ for $\boldsymbol{F}_s^{\mathrm{I,II}}(\boldsymbol{s}, v, z)$. The proof is by a fixed-point argument. Define a set-valued map $\Phi : \Omega :\to \Omega$ by letting, for each $(\boldsymbol{s}^0, \boldsymbol{g}^0) \in \Omega$, $\Phi(\boldsymbol{s}^0, \boldsymbol{g}^0)$ be the set of $(\boldsymbol{s}, \boldsymbol{g})$ for which there exist $(v, z, \boldsymbol{\eta}, \varphi)$ such that $(v, z, \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{\eta}, \varphi)$ solves the following monotone mixed LCP:

$$\begin{array}{lll} 0 & \le v & \perp \\ 0 & \le z & \perp \\ 0 & \le \boldsymbol{s} & \perp \\ 0 & \le \boldsymbol{g} & \perp \\ 0 & \le \boldsymbol{\eta} & \perp \\ \text{free} & \;\;\varphi & \end{array} \begin{pmatrix} q_v \\ q_z \\ \boldsymbol{q}_s \\ \boldsymbol{q}_g \\ \textbf{CAP} \\ 0 \end{pmatrix} + \boldsymbol{M} \begin{pmatrix} v \\ z \\ \boldsymbol{s} \\ \boldsymbol{g} \\ \boldsymbol{\eta} \\ \varphi \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{F}_s^{\mathrm{I,II}}(\boldsymbol{s}^0, \boldsymbol{g}^0) \\ \mathrm{vec}(c'_{fi}(g_{fi}^0)) \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \ge 0 \\ \ge 0 \\ \ge 0 \\ \ge 0 \\ \ge 0 \\ = 0. \end{array} \tag{13}$$

Provided that Φ satisfies two properties:

- $\Phi(\boldsymbol{s}, \boldsymbol{g})$ is a nonempty and convex subset of Ω,
- Φ is a closed set-valued map; i.e., for every sequence $\{(\boldsymbol{s}^k, \boldsymbol{g}^k)\}$ converging to $(\boldsymbol{s}^\infty, \boldsymbol{g}^\infty)$ and for any sequence $\{(\bar{\boldsymbol{s}}^k, \bar{\boldsymbol{g}}^k)\}$ converging to $(\bar{\boldsymbol{s}}^\infty, \bar{\boldsymbol{g}}^\infty)$ with $(\bar{\boldsymbol{s}}^k, \bar{\boldsymbol{g}}^k) \in \Phi(\boldsymbol{s}^k, \boldsymbol{g}^k)$ for every k, we have $(\bar{\boldsymbol{s}}^\infty, \bar{\boldsymbol{g}}^\infty) \in \Phi(\boldsymbol{s}^\infty, \boldsymbol{g}^\infty)$,

Kakutani's fixed-point theorem then implies that Φ has a fixed point; i.e., there exists $(\boldsymbol{s}^*, \boldsymbol{g}^*) \in \Phi(\boldsymbol{s}^*, \boldsymbol{g}^*)$. Such a pair along with the auxiliary variables $(v, z, \boldsymbol{\eta}, \varphi)$ then provides a solution to the respective model. To show the first property, let $(\boldsymbol{s}^0, \boldsymbol{g}^0) \in \Omega$ be given. Since $\boldsymbol{M}$ is positive semidefinite, for $\Phi(\boldsymbol{s}^0, \boldsymbol{g}^0)$ to be nonempty, it suffices for the the LCP (13) to be feasible. By the definition of Ω, there exists (v^0, z^0) satisfying

$$\begin{array}{l} 0 \\ 0 \end{array} \begin{array}{l} \leq \\ \leq \end{array} \begin{pmatrix} v^0 \\ z^0 \end{pmatrix}, \quad \begin{pmatrix} q_v \\ q_z \end{pmatrix} + \begin{bmatrix} \boldsymbol{M}_v & \boldsymbol{M}_{vz} \\ \boldsymbol{M}_{zv} & \boldsymbol{M}_z \end{bmatrix} \begin{pmatrix} v^0 \\ z^0 \end{pmatrix} + \begin{bmatrix} \boldsymbol{M}_{vs} & \boldsymbol{M}_{vg} \\ \boldsymbol{M}_{zs} & \boldsymbol{M}_{zg} \end{bmatrix} \begin{pmatrix} \boldsymbol{s}^0 \\ \boldsymbol{g}^0 \end{pmatrix} \begin{array}{l} \geq 0 \\ \geq 0. \end{array}$$

It is clear that we can choose $(\boldsymbol{\eta}^0, \varphi^0)$ such that $(v^0, z^0, \boldsymbol{s}^0, \boldsymbol{g}^0, \boldsymbol{\eta}^0, \varphi^0)$ is feasible to (13). Since $\boldsymbol{M}$ is positive semidefinite, the solution set of this LCP is convex; thus so is $\Phi(\boldsymbol{s}^0, \boldsymbol{g}^0)$.

To show the closedness of the map Φ, let the sequence $\{(\boldsymbol{s}^k, \boldsymbol{g}^k)\}$ converge to $(\boldsymbol{s}^\infty, \boldsymbol{g}^\infty)$ and let the sequence $\{(\bar{\boldsymbol{s}}^k, \bar{\boldsymbol{g}}^k)\}$ converge to $(\bar{\boldsymbol{s}}^\infty, \bar{\boldsymbol{g}}^\infty)$ such that $(\bar{\boldsymbol{s}}^k, \bar{\boldsymbol{g}}^k) \in \Phi(\boldsymbol{s}^k, \boldsymbol{g}^k)$ for every k. For each k, there exists $(v^k, z^k, \boldsymbol{\eta}^k, \varphi^k)$ such that $(v^k, z^k, \bar{\boldsymbol{s}}^k, \bar{\boldsymbol{g}}^k, \boldsymbol{\eta}^k, \varphi^k)$ solves the LCP (13) with $(\boldsymbol{s}^0, \boldsymbol{g}^0)$ replaced by $(\boldsymbol{s}^k, \boldsymbol{g}^k)$. To complete the proof, it suffices to demonstrate that we can choose a bounded sequence $\{(v^k, z^k, \boldsymbol{\eta}^k, \varphi^k)\}$ with this property. The choice of the latter sequence is based on a complementary cone argument, which in turn utilizes the fact that there are only finitely many such cones corresponding to a given matrix that defines an LCP. The argument is fairly standard in LCP theory [8] and is omitted. □

Uniqueness of solutions Let $(v^k, z^k, \boldsymbol{s}^k, \boldsymbol{g}^k, \boldsymbol{\eta}^k, \varphi^k)$, $k = 1, 2$, be two solutions of the exogenous-arbitrage model under the variable-slope, fixed-intercept conjecture. Let

$$p^k = \boldsymbol{A} (Q^0 - e^0 - S^k) + [I - \boldsymbol{A}\text{Diag}(Q_i^0/P_i^0)] [(DE^a)^T v^k + H^T z^k]$$

be the corresponding price vectors. We have, by Proposition 1 and the symmetry and positive semidefiniteness of $\boldsymbol{A}$,

$$\begin{aligned} &\begin{pmatrix} v^1 - v^2 \\ z^1 - v^2 \\ \boldsymbol{s}^1 - \boldsymbol{s}^2 \\ \boldsymbol{g}^1 - \boldsymbol{g}^2 \\ \boldsymbol{\eta}^1 - \boldsymbol{\eta}^2 \\ \varphi^1 - \varphi^2 \end{pmatrix}^T \boldsymbol{M} \begin{pmatrix} v^1 - v^2 \\ z^1 - v^2 \\ \boldsymbol{s}^1 - \boldsymbol{s}^2 \\ \boldsymbol{g}^1 - \boldsymbol{g}^2 \\ \boldsymbol{\eta}^1 - \boldsymbol{\eta}^2 \\ \varphi^1 - \varphi^2 \end{pmatrix} \\ &= \begin{pmatrix} v^1 - v^2 \\ z^1 - v^2 \end{pmatrix} \begin{bmatrix} \boldsymbol{M}_v & \boldsymbol{M}_{vz} \\ \boldsymbol{M}_{zv} & \boldsymbol{M}_z \end{bmatrix} \begin{pmatrix} v^1 - v^2 \\ z^1 - v^2 \end{pmatrix} + \begin{pmatrix} \boldsymbol{s}^1 - \boldsymbol{s}^2 \\ \boldsymbol{g}^1 - \boldsymbol{g}^2 \end{pmatrix} \begin{bmatrix} \boldsymbol{M}_s & \boldsymbol{M}_{sg} \\ \boldsymbol{M}_{gs} & \boldsymbol{M}_g \end{bmatrix} \begin{pmatrix} \boldsymbol{s}^1 - \boldsymbol{s}^2 \\ \boldsymbol{g}^1 - \boldsymbol{g}^2 \end{pmatrix} \\ &\geq \delta \, \| \, [\, I - \boldsymbol{A}\text{Diag}(Q_i^0/P_i^0) \,] \, [\, (DE^a \,)^T (\, v^1 - v^2 \,) + H^T (\, z^1 - z^2 \,) \,] \, \|^2 + \\ &\qquad (\, S^1 - S^2 \,)^T \boldsymbol{A} (\, S^1 - S^2 \,) + \\ &\sum_{f \in \mathcal{F}} [B^f (\boldsymbol{s}^1 - \boldsymbol{s}^2)^f + E^f (\boldsymbol{g}^1 - \boldsymbol{g}^2)^f]^T \text{Diag}(\sigma_{fi}) [B^f (\boldsymbol{s}^1 - \boldsymbol{s}^2)^f + E^f (\boldsymbol{g}^1 - \boldsymbol{g}^2)^f] \\ &\geq \delta \, \| \, [\, I - \boldsymbol{A}\text{Diag}(Q_i^0/P_i^0) \,] \, [\, (DE^a \,)^T (\, v^1 - v^2 \,) + H^T (\, z^1 - z^2 \,) \,] \, \|^2 + \\ &\qquad \xi \, \| \, \boldsymbol{A} (\, S^1 - S^2 \,) \|^2 + \sum_{f \in \mathcal{F}} \min_{1 \leq i \leq n} \sigma_{fi} \, \| \, B^f (\boldsymbol{s}^1 - \boldsymbol{s}^2)^f + E^f (\boldsymbol{g}^1 - \boldsymbol{g}^2)^f \, \|^2, \end{aligned}$$

where ξ is the reciprocal of the largest eigenvalue of $\boldsymbol{A}$; moreover,

$$
\begin{aligned}
&\begin{pmatrix} \boldsymbol{s}^1 - \boldsymbol{s}^2 \\ \boldsymbol{g}^1 - \boldsymbol{g}^2 \end{pmatrix}^T \begin{pmatrix} \boldsymbol{F}_s^{\mathrm{II}}(\boldsymbol{s}^1, v^1, z^1) - \boldsymbol{F}_s^{\mathrm{II}}(\boldsymbol{s}^2, v^2, z^2) \\ \mathrm{vec}(c'_{fi}(g^1_{fi})) - \mathrm{vec}(c'_{fi}(g^1_{fi})) \end{pmatrix} \\
&= \sum_{f \in \mathcal{F}} \sum_{i=1}^{n} (\, s^1_{fi} - s^2_{fi} \,) \left\{ \left[\frac{Q_i^0}{P_i^0} + \frac{s^1_{-fi}}{p_i^1 - \alpha_{fi}} \right]^{-1} s^1_{fi} - \left[\frac{Q_i^0}{P_i^0} + \frac{s^2_{-fi}}{p_i^2 - \alpha_{fi}} \right]^{-1} s^2_{fi} \right\} \\
&\quad + \sum_{f \in \mathcal{F}} \sum_{i=1}^{n} (\, g^1_{fi} - g^2_{fi} \,) \, [c'_{fi}(g^1_{fi})! - c'_{fi}(g^2_{fi})] \\
&= \sum_{f \in \mathcal{F}} \sum_{i=1}^{n} \left[\frac{Q_i^0}{P_i^0} + \frac{s^1_{-fi}}{p_i^1 - \alpha_{fi}} |! \right]^{-1} (s^1_{fi} - s^2_{fi})^2 \\
&\quad + \sum_{f \in \mathcal{F}} \sum_{i=1}^{n} (\, s^1_{fi} - s^2_{fi} \,) \left\{ \left[\frac{Q_i^0}{P_i^0} + \frac{s^1_{-fi}}{p_i^1 - \alpha_{fi}} \right]^{-1} - \left[\frac{Q_i^0}{P_i^0} + \frac{s^2_{-fi}}{p_i^2 - \alpha_{fi}} \right]^{-1} \right\} s^2_{fi} \\
&\quad + \sum_{f \in \mathcal{F}} \sum_{i=1}^{n} (\, g^1_{fi} - g^2_{fi} \,) \, [\, c'_{fi}(g^1_{fi}) - c'_{fi}(g^2_{fi}) \,].
\end{aligned}
$$

We have

$$
\begin{aligned}
&\left[\frac{Q_i^0}{P_i^0} + \frac{s^1_{-fi}}{p_i^1 - \alpha_{fi}} \right]^{-1} - \left[\frac{Q_i^0}{P_i^0} + \frac{s^2_{-fi}}{p_i^2 - \alpha_{fi}} \right]^{-1} = \\
&\quad \left[\frac{Q_i^0}{P_i^0} + \frac{s^1_{-fi}}{p_i^1 - \alpha_{fi}} \right]^{-1} \left[\frac{Q_i^0}{P_i^0} + \frac{s^2_{-fi}}{p_i^2 - \alpha_{fi}} \right]^{-1} \left[\frac{s^2_{-fi}}{p_i^2 - \alpha_{fi}} - \frac{s^1_{-fi}}{p_i^1 - \alpha_{fi}} \right].
\end{aligned}
$$

By Lemma 1, we have

$$
\begin{aligned}
\left| \frac{s^1_{-fi}}{p_i^1 - \alpha_{fi}} - \frac{s^2_{-fi}}{p_i^2 - \alpha_{fi}} \right| \quad &\le \tau(\alpha_{fi}) \, [\, |\, s^1_{-fi} - s^2_{-fi} \,| + |\, p_i^2 - p_i^2 \,| \,] \\
&\le \tau(\alpha_{fi}) \Bigg[\sum_{h \ne f, h \in \mathcal{F}} |\, s^1_{hi} - s^2_{hi} \,| + \| \, \boldsymbol{A}(\, S^1 - S^2 \,) \, \| + \\
&\| \, [\, I - \boldsymbol{A} \mathrm{Diag}(Q_i^0 / P_i^0) \,] \, [\, (DE^a)^T (\, v^1 - v^2 \,) + H^T (\, z^1 - z^2 \,) \,] \, \| \Bigg],
\end{aligned}
$$

where

$$
\lim_{|\alpha| \to \infty} \tau(\alpha) \; = \; 0.
$$

Consequently,

$$\begin{pmatrix} \boldsymbol{s}^1 - \boldsymbol{s}^2 \\ \boldsymbol{g}^1 - \boldsymbol{g}^2 \end{pmatrix}^T \begin{pmatrix} \boldsymbol{F}_s^{\mathrm{II}}(\boldsymbol{s}^1, v^1, z^1) - \boldsymbol{F}_s^{\mathrm{II}}(\boldsymbol{s}^2, v^2, z^2) \\ \mathrm{vec}(c'_{fi}(g^1_{fi})) - \mathrm{vec}(c'_{fi}(g^1_{fi})) \end{pmatrix}$$
$$\geq \sum_{f\in\mathcal{F}} \sum_{i=1}^{n} \left[\frac{Q_i^0}{P_i^0} + \frac{s^1_{-fi}}{p_i^1 - \alpha_{fi}} \right]^{-1} (s^1_{fi} - s^2_{fi})^2 -$$
$$\sum_{f\in\mathcal{F}} \sum_{i=1}^{n} \tau(\alpha_{fi}) \,|\, s^1_{fi} - s^2_{fi} \,|\, \Bigg[\sum_{h\neq f, h\in\mathcal{F}} |\, s^1_{hi} - s^2_{hi} \,| + \|\, \boldsymbol{A}(S^1 - S^2) \,\| +$$
$$\|\, [\, I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0) \,] \,[\, (DE^a)^T (v^1 - v^2) + H^T (z^1 - z^2) \,] \,\| \Bigg] +$$
$$\sum_{f\in\mathcal{F}} \sum_{i=1}^{n} (g^1_{fi} - g^2_{fi}) \,[\, c'_{fi}(g^1_{fi}) - c'_{fi}(g^2_{fi}) \,],$$

Based on the above bounds, we can establish the following result that identifies certain uniqueness properties of the solutions to the exogenous-arbitrage model.

Theorem 2. Assume that G has full row rank and that each cost function $c_{fi}(g_{fi})$ is convex and continuously differentiable. Suppose that the set Ω is nonempty. There exists a constant α' such that if $|\alpha_{fi}| > \alpha'$ for all $f \in \mathcal{F}$ and all $i = 1, \cdots, n$, the following quantities are unique in any equilibrium solution to the variable-slope, fixed-intercept, exogenous-arbitrage model :

(a) the sales s_{fi} for all $f \in \mathcal{F}$ and $i \in \mathcal{N}$,
(b) the prices p_i for all $i \in \mathcal{N}$,
(c) the arbitrage quantities a_i for all $i \in \mathcal{N}$,
(d) the resource usage r_{fj} and r_{aj} for all $f \in \mathcal{F}$ and $j = 1, \ldots, m$,
(e) the marginal generation costs $c'_{fi}(g_{fi})$ for each $f \in \mathcal{F}$ and $i \in \mathcal{N}$,
(f) the total generation $\sum_{i=1}^{n} g_{fi}$ for each $f \in \mathcal{F}$.

Moreover, if each c'_{fi} is strictly increasing, then the amounts generated g_{fi} are also unique.

Proof. Continuing the above derivation, we note that, by complementarity,

$$\begin{pmatrix} v^1 - v^2 \\ z^1 - v^2 \\ \boldsymbol{s}^1 - \boldsymbol{s}^2 \\ \boldsymbol{g}^1 - \boldsymbol{g}^2 \\ \boldsymbol{\eta}^1 - \boldsymbol{\eta}^2 \\ \varphi^1 - \varphi^2 \end{pmatrix}^T \boldsymbol{M} \begin{pmatrix} v^1 - v^2 \\ z^1 - v^2 \\ \boldsymbol{s}^1 - \boldsymbol{s}^2 \\ \boldsymbol{g}^1 - \boldsymbol{g}^2 \\ \boldsymbol{\eta}^1 - \boldsymbol{\eta}^2 \\ \varphi^1 - \varphi^2 \end{pmatrix}$$
$$+ \begin{pmatrix} \boldsymbol{s}^1 - \boldsymbol{s}^2 \\ \boldsymbol{g}^1 - \boldsymbol{g}^2 \end{pmatrix}^T \begin{pmatrix} \boldsymbol{F}_s^{\mathrm{II}}(\boldsymbol{s}^1, v^1, z^1) - \boldsymbol{F}_s^{\mathrm{II}}(\boldsymbol{s}^2, v^2, z^2) \\ \mathrm{vec}(c'_{fi}(g^1_{fi})) - \mathrm{vec}(c'_{fi}(g^1_{fi})) \end{pmatrix} \leq 0.$$

The above derivations show that the sum of the two inner products in the left-hand side is bounded below by a positive definite quadratic form in the quantities:

$$| s^1_{fi} - s^2_{fi} |, \qquad \| \boldsymbol{A}(S^1 - S^2) \|,$$

$$\| [I - \boldsymbol{A}\mathrm{Diag}(Q^0_i / P^0_i)] [(DE^a)^T (v^1 - v^2) + H^T (z^1 - z^2)] \|$$

and

$$\| B^f (\boldsymbol{s}^1 - \boldsymbol{s}^2)^f + E^f (\boldsymbol{g}^1 - \boldsymbol{g}^2)^f \|,$$

provided that the $|\alpha_{fi}|$ are sufficiently large. Consequently, there exists α' such that if $|\alpha_{fi}| > \alpha'$ for all $f \in \mathcal{F}$ and $i = 1, \cdots, n$, then

$$[I - \boldsymbol{A}\mathrm{Diag}(Q^0_i / P^0_i)] [(DE^a)^T (v^1 - v^2) + H^T (z^1 - z^2)] = 0$$

$$B^f (\boldsymbol{s}^1 - \boldsymbol{s}^2)^f + E^f (\boldsymbol{g}^1 - \boldsymbol{g}^2)^f = 0, \qquad \forall f \in \mathcal{F},$$

and for all $f \in \mathcal{F}$ and $i = 1, \cdots, n$,

$$s^1_{fi} = s^2_{fi} \quad \text{and} \quad (g^1_{fi} - g^2_{fi}) [c'_{fi}(g^1_{fi}) - c'_{fi}(g^2_{fi})] = 0.$$

Since, for $k = 1, 2$,

$$p^k = \boldsymbol{A} (Q^0 - e^0 - S^k) + [I - \boldsymbol{A}\mathrm{Diag}(Q^0_i / P^0_i)] [(DE^a)^T v^k + H^T z^k],$$

$$(r^k)^f = B^f (\boldsymbol{s}^k)^f + E^f (\boldsymbol{g}^k)^f - \omega^f, \qquad \forall f \in \mathcal{F},$$

$$a^k = Q^0 - S^k - \mathrm{Diag}(Q^0_i / P^0_i)\, p^k,$$

$$(r^k)^a = E^a a^k - \omega^a,$$

and

$$\sum_{i=1}^n g^k_{fi} = \sum_{i=1}^n s^k_{fi}, \qquad \forall f \in \mathcal{F},$$

the statements (a)–(f) all follow readily. The last assertion about the uniqueness of the amounts of generation is also clear. □

Due to the close connection between the endogenous-arbitrage and the exogenous-arbitrage model, it is interesting to explore the relationship between the solutions to the two models. The next result identifies a simple condition under which a solution to one model will provide a solution to the other model. See Theorem 4 for an application of the result.

Proposition 4. Assume that G has full row rank. If $(v, z, \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{\eta}, \varphi)$ is a solution to the NCP (10) or (11) and if $\mathrm{Diag}(\tilde{\beta}_{fi}(p_i, s_{-fi}))^{-1} s^f$ is the range of G^T for all $f \in \mathcal{F}$, then $(v, z, \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{\eta}, \varphi)$ is a solution to both NCPs.

Proof. This is easy because if $\mathrm{Diag}(\tilde{\beta}_{fi}(p_i, s_{-fi}))^{-1} s^f$ is the range of G^T, then

$$\mathrm{Diag}(\tilde{\beta}_{fi}(p_i, s_{-fi}))^{-1} s^f \;=\; A^f(p, s^{-f}) s^f.$$

Hence $\boldsymbol{F}_s^{\mathrm{I}}(\boldsymbol{s}, v, z) = \boldsymbol{F}_s^{\mathrm{II}}(\boldsymbol{s}, v, z)$ and the desired conclusion follows readily. □

The practical importance of this equivalence result is that whichever model is easiest to compute can be used.

3.2. *The fixed-slope, variable-intercept conjecture*

Unlike the exogenous-arbitrage model, uniqueness results are not readily available for the solutions to the endogenous-arbitrage model under the variable-slope, fixed-intercept conjecture (4) (but see the next subsection for a special model). Nevertheless, a subset of the solution uniqueness properties remain valid in the endogenous-arbitrage model under the fixed-slope, variable-intercept conjecture. In what follows, we write

$$\tilde{\alpha}_{fi} \;\equiv\; \frac{Q_i^0}{P_i^0} + \alpha_{fi},$$

which is equal to $\tilde{\beta}_{fi}(x, y)$ when $\beta_{fi}(x, y)$ is the constant α_{fi}, which we assume is positive.

Theorem 3. Assume that G has full row rank and that each cost function $c_{fi}(g_{fi})$ is convex and continuously differentiable. Suppose that the set Ω is nonempty. In the case of the fixed-slope, variable-intercept conjecture (3), equilibrium solutions exist for both the endogenous- and exogenous-arbitrage models. Moreover, for any such solution, the following quantities are unique in the respective models:

(a) the sales s_{fi} for all $f \in \mathcal{F}$ and $i = 1, \ldots, n$ in the exogenous model;
(b) the vector Gs^f for all $f \in \mathcal{F}$ in both models,
(c) the prices p_i for all $i \in \mathcal{N}$ in both models,
(d) the arbitrage quantities a_i for all $i \in \mathcal{N}$ and the arbitrage resource usage r_{aj} for $j = 1, \ldots, m$ in the exogenous model,
(e) the sum of arbitrage and firms' sales $a_i + \sum_{f \in \mathcal{F}} s_{fi}$ for all $i \in \mathcal{N}$ in both models,
(f) the firm's resource usage r_{fj} for each $f \in \mathcal{F}$ and $j = 1, \ldots, m$ in both models,
(g) the marginal generation cost $c'_{fi}(g_{fi})$ for each $f \in \mathcal{F}$ and $i \in \mathcal{N}$ in both models, and
(h) the total generation $\sum_{i=1}^{n} g_{fi}$ for each $f \in \mathcal{F}$ in the exogenous model.

Finally, if each c'_{fi} is strictly increasing, then the amounts generated g_{fi} are unique in both models.

Proof. We focus on the endogenous-arbitrage model because the results for the exogenous-arbitrage model follow easily from a similar analysis to that in the last subsection. In the fixed-slope, variable-intercept model, we have

$$\boldsymbol{F}^{\mathrm{I}}_s(\boldsymbol{s}, v, z) \;=\; \mathrm{Diag}(\, G^T\,(\,G\mathrm{Diag}(\tilde{\alpha}_{fi})G^T\,)^{-1}G\,)\boldsymbol{s}$$

which is a monotone, linear function of $\boldsymbol{s}$. From this observation, it follows from the same proof as in Theorem 2 that the following vectors are unique,:

$$Gs^f, \quad \boldsymbol{A}S, \quad [\,I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)\,]\,[\,(\,DE^a)^T v + H^T z\,],$$

and $B^f s^f + E^f g^f$. Again, since

$$p \;=\; \boldsymbol{A}\,(\,Q^0 - e^0 - S\,) + [\,I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)\,]\,[\,(\,DE^a)^T v + H^T z\,],$$

$$r^f \;=\; B^f s^f + E^f g^f - \omega^f, \quad \forall\, f \in \mathcal{F},$$

and

$$a + S \;=\; Q^0 - \mathrm{Diag}(Q_i^0/P_i^0)\,p,$$

it follows that statements (b), (c), (e) and (f) hold. Finally, as in Theorem 2, statement (g) and the last assertion also hold for the endogenous-arbitrage model. □

3.3. *A special model*

An important special case of the models presented in the last two sections occurs under the following specifications: the resources utilized by each firm are only for production and not for sales (thus $B^f = 0$ for all $f \in \mathcal{F}$), the arbitrage does not use input resources (so $E^a = 0$), and the arbitrage constraint is $\sum_{i=1}^{n} a_i = 0$ (thus G is the row vector of all ones). This is the case, for instance, with emissions allowances, which are only required for electricity generation. In this circumstance, some uniqueness properties of the solutions to the endogenous-arbitrage model with the variable-slope, fixed-intercept conjecture (4) can be derived and further connections between the solutions to the endogenous-arbitrage model and the exogenous-arbitrage model can be established. Before stating these results, we note that under these specifications, the matrices $\boldsymbol{A}$ and $A^f(p, s^{-f})$ both become special rank-one matrices given by

$$\boldsymbol{A} \;=\; \left(\sum_{i=1}^{n} \frac{Q_i^0}{P_i^0} \right)^{-1} \boldsymbol{E}_n,$$

and for all $f \in \mathcal{F}$,

$$A^f(p, s^{-f}) \;=\; \left[\, \sum_{i=1}^{n} \left(\frac{Q_i^0}{P_i^0} + \beta_{fi}(p_i, s_{-fi}) \right) \right]^{-1} \boldsymbol{E}_n,$$

where $\boldsymbol{E}_n$ is the square matrix of all ones of order n. Exploiting this observation, we state the following additional result for the special model.

The following theorem extends the results in the previous papers [23, 28].

Theorem 4. For the special model described above, the following statements hold.

(a) If $(v, z, \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{\eta}, \varphi)$ is a solution to the NCP (11), then for all $f \in \mathcal{F}$,

$$\frac{s_{f1}}{\dfrac{Q_1^0}{P_1^0} + \beta_{f1}(p_1, s_{-f1})} = \frac{s_{fi}}{\dfrac{Q_i^0}{P_i^0} + \beta_{fi}(p_i, s_{-fi})}, \quad \forall\, i = 1, \ldots, n.$$

Consequently, $(v, z, \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{\eta}, \varphi)$ is also a solution to the NCP (10); thus any solution to the exogenous-arbitrage model is a solution to the endogenous-arbitrage model, under either the variable-slope, fixed-intercept or the fixed-slope, variable-intercept conjecture.

(b) Conversely, if $(v, z, \boldsymbol{s}^{\mathrm{I}}, \boldsymbol{g}, \boldsymbol{\eta}, \varphi)$ is a solution to the NCP (10) under the fixed-slope, variable-intercept conjecture, by defining, for all $f \in \mathcal{F}$ and $i = 1, \ldots, n$,

$$s_{fi}^{\mathrm{II}} \equiv \frac{\dfrac{Q_i^0}{P_i^0} + \alpha_{fi}}{\displaystyle\sum_{j=1}^{n} \left(\frac{Q_j^0}{P_j^0} + \alpha_{fj} \right)} \sum_{j=1}^{n} s_{fj}^{\mathrm{I}},$$

then $(v, z, \boldsymbol{s}^{\mathrm{II}}, \boldsymbol{g}, \boldsymbol{\eta}, \varphi)$ solves the NCP (11) under the fixed-slope, variable-intercept conjecture.

(c) The following equilibrium quantities in the endogenous-arbitrage model under the fixed-slope, variable-intercept conjecture are equal to the respective (unique) quantities in the exogenous-arbitrage model:

(i) the firms' total sales $\displaystyle\sum_{i=1}^{n} s_{fi}$ (and hence the total generation $\displaystyle\sum_{i=1}^{n} g_{fi}$) for each $f \in \mathcal{F}$;

(ii) the prices p_i for all $i \in \mathcal{N}$;

(iii) the firm's resource use r_{fj} for each $f \in \mathcal{F}$ and $j = 1, \ldots, m$;

(iv) the marginal generation cost $c'_{fi}(g_{fi})$ for each $f \in \mathcal{F}$ and $i \in \mathcal{N}$.

Proof. The sole difference between the endogenous-arbitrage model and the exogenous-arbitrage model is in the functions $\boldsymbol{F}_s^{\mathrm{I,II}}$, which affect only the expression that is complementary to the variables s_{fi}. Hence, we can write out these complementarity relations in the two models, using the superscripts I and II to distinguish the variables in the former and the latter models, respectively: for all $f \in \mathcal{F}$ and $i = 1, \ldots, n$,

$$0 \le s_{fi}^{\mathrm{I}} \perp \sum_{j=1}^{n} \left[e_j^0 - Q_j^0 + \sum_{h \in \mathcal{F}} s_{hj}^{\mathrm{I}} \right] + \frac{\left(\displaystyle\sum_{j=1}^{n} \frac{Q_j^0}{P_j^0} \right) \left(\displaystyle\sum_{j=1}^{n} s_{fj}^{\mathrm{I}} \right)}{\displaystyle\sum_{j=1}^{n} \left(\frac{Q_j^0}{P_j^0} + \beta_{fj}(p_j^{\mathrm{I}}, s_{-fj}^{\mathrm{I}}) \right)} \tag{14}$$

$$+ \sum_{j=1}^{n} \frac{Q_j^0}{P_j^0} \left[(H^T z^{\mathrm{I}})_j + \varphi_f^{\mathrm{I}} \right] \ge 0$$

and

$$0 \leq s^{\mathrm{II}}_{fi} \perp \sum_{j=1}^{n} \left[e^0_j - Q^0_j + \sum_{h \in \mathcal{F}} s^{\mathrm{II}}_{hj} \right] + \frac{\left(\sum_{j=1}^{n} \dfrac{Q^0_j}{P^0_j} \right) s^{\mathrm{II}}_{fi}}{\dfrac{Q^0_i}{P^0_i} + \beta_{fi}(p^{\mathrm{II}}_i, s^{\mathrm{II}}_{-fi})}$$
$$+ \sum_{j=1}^{n} \frac{Q^0_j}{P^0_j} \left[(H^T z^{\mathrm{II}})_j + \varphi^{\mathrm{II}}_f \right] \geq 0.$$

Let $(\boldsymbol{s}^{\mathrm{II}}, z^{\mathrm{II}}, \varphi^{\mathrm{II}})$ be a triple satisfying the latter complementarity condition for all f and i. Fix f; we claim that

$$\frac{s^{\mathrm{II}}_{f1}}{\dfrac{Q^0_1}{P^0_1} + \beta_{f1}(p^{\mathrm{II}}_1, s^{\mathrm{II}}_{-f1})} \leq \frac{s^{\mathrm{II}}_{fi}}{\dfrac{Q^0_i}{P^0_i} + \beta_{fi}(p^{\mathrm{II}}_i, s^{\mathrm{II}}_{-fi})}, \quad \forall i > 1. \tag{15}$$

This inequality is clearly valid if $s^{\mathrm{II}}_{f1} = 0$. If $s^{\mathrm{II}}_{f1} > 0$, then

$$\sum_{j=1}^{n} \left[e^0_j - Q^0_j + \sum_{h \in \mathcal{F}} s^{\mathrm{II}}_{hj} \right] + \frac{\left(\sum_{j=1}^{n} \dfrac{Q^0_j}{P^0_j} \right) s^{\mathrm{II}}_{f1}}{\dfrac{Q^0_1}{P^0_1} + \beta_{f1}(p^{\mathrm{II}}_1, s^{\mathrm{II}}_{-f1})} + \sum_{j=1}^{n} \frac{Q^0_j}{P^0_j} \left[(H^T z^{\mathrm{II}})_j + \varphi^{\mathrm{II}}_f \right]$$
$$= 0 \leq$$
$$\sum_{j=1}^{n} \left[e^0_j - Q^0_j + \sum_{h \in \mathcal{F}} s^{\mathrm{II}}_{hj} \right] + \frac{\left(\sum_{j=1}^{n} \dfrac{Q^0_j}{P^0_j} \right) s^{\mathrm{II}}_{fi}}{\dfrac{Q^0_i}{P^0_i} + \beta_{fi}(p^{\mathrm{II}}_i, s^{\mathrm{II}}_{-fi})} + \sum_{j=1}^{n} \frac{Q^0_j}{P^0_j} \left[(H^T z^{\mathrm{II}})_j + \varphi^{\mathrm{II}}_f \right],$$

which clearly implies the inequality (15). Reversing the role of the index 1 with the index i, we deduce that the inequality must hold as an equality. Thus the first part of statement (a) is established. From this, it follows that

$$\frac{s^{\mathrm{II}}_{fi}}{\dfrac{Q^0_i}{P^0_i} + \beta_{fi}(p^{\mathrm{II}}_i, s^{\mathrm{II}}_{-fi})} = \frac{\sum_{j=1}^{n} s^{\mathrm{II}}_{fj}}{\sum_{j=1}^{n} \left[\dfrac{Q^0_j}{P^0_j} + \beta_{fj}(p^{\mathrm{II}}_j, s^{\mathrm{II}}_{-fj}) \right]}, \quad \forall i = 1, \ldots, n.$$

Consequently, $(\boldsymbol{s}^{\mathrm{II}}, z^{\mathrm{II}}, \varphi^{\mathrm{II}})$ satisfies (14). This establishes statement (a).

To prove statement (b), note that for each f, the expression that is complementary to s^{I}_{fi} in (14) depends only on f but is independent of i. Also recall that $\beta_{fi}(x, y)$ is equal to the constant α_{fi} in this case. Consequently, if $s^{\mathrm{I}}_{fi} > 0$ for some i, then

$$\sum_{j=1}^{n}\left[e^0_j - Q^0_j + \sum_{h\in\mathcal{F}} s^{\mathrm{I}}_{hj} \right] + \frac{\left(\displaystyle\sum_{j=1}^{n} \frac{Q^0_j}{P^0_j} \right) \left(\displaystyle\sum_{j=1}^{n} s^{\mathrm{I}}_{fj} \right)}{\displaystyle\sum_{j=1}^{n} \left(\frac{Q^0_j}{P^0_j} + \alpha_{fj} \right)}$$
$$+ \sum_{j=1}^{n} \frac{Q^0_j}{P^0_j} [\,(H^T z^{\mathrm{I}})_j + \varphi^{\mathrm{I}}_f\,] = 0.$$

Now let $(v, z, \boldsymbol{s}^{\mathrm{I}}, \boldsymbol{g}, \boldsymbol{\eta}, \varphi)$ be a solution to the NCP (10) under the fixed-slope, variable-intercept conjecture. Let $\boldsymbol{s}^{\mathrm{II}}$ be defined in part (b). Clearly,

$$\sum_{i=1}^{n} s^{\mathrm{II}}_{fi} = \sum_{i=1}^{n} s^{\mathrm{I}}_{fi}, \quad \forall f \in \mathcal{F}.$$

Fix f and consider two situations: $\displaystyle\sum_{i=1}^{n} s^{\mathrm{I}}_{fi} = 0$ and $\displaystyle\sum_{i=1}^{n} s^{\mathrm{I}}_{fi} > 0$. In the former situation, we have $s^{\mathrm{II}}_{fi} = s^{\mathrm{I}}_{fi} = 0$ for all i. In the latter situation, we have

$$\sum_{j=1}^{n}\left[e^0_j - Q^0_j + \sum_{h\in\mathcal{F}} s^{\mathrm{II}}_{hj} \right] + \frac{\left(\displaystyle\sum_{j=1}^{n} \frac{Q^0_j}{P^0_j} \right) s^{\mathrm{II}}_{fi}}{\dfrac{Q^0_i}{P^0_i} + \alpha_{fi}} + \sum_{j=1}^{n} \frac{Q^0_j}{P^0_j} [\,(H^T z^{\mathrm{I}})_j + \varphi^{\mathrm{I}}_f\,] = 0.$$

Thus $(v, z, \boldsymbol{s}^{\mathrm{II}}, \boldsymbol{g}, \boldsymbol{\eta}, \varphi)$ solves (11) under the fixed-slope, variable-intercept conjecture.

By part (a) and the uniqueness in Theorem 3, the respective quantities in (i)–(iv) must be equal in the exogenous-arbitrage and the endogenous-arbitrage model. □

3.4. The nonemptiness of Ω

To complete the analysis of the models, we consider the issue of nonemptiness of the polyhedron Ω. Since $\bar{\boldsymbol{M}}$ is a symmetric positive semidefinite matrix,

$$\begin{pmatrix} q_v \\ q_z \end{pmatrix} + \begin{bmatrix} \boldsymbol{M}_{vs} & \boldsymbol{M}_{vg} \\ \boldsymbol{M}_{zs} & \boldsymbol{M}_{zg} \end{bmatrix} \begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{g} \end{pmatrix} \in \mathcal{R}(\bar{\boldsymbol{M}}) \tag{16}$$

if and only if there exists (v, z) such that

$$\begin{pmatrix} q_v \\ q_z \end{pmatrix} + \begin{bmatrix} \boldsymbol{M}_{vs} & \boldsymbol{M}_{vg} \\ \boldsymbol{M}_{zs} & \boldsymbol{M}_{zg} \end{bmatrix} \begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{g} \end{pmatrix} + \begin{bmatrix} \boldsymbol{M}_{v} & \boldsymbol{M}_{vz} \\ \boldsymbol{M}_{zv} & \boldsymbol{M}_{z} \end{bmatrix} \begin{pmatrix} v \\ z \end{pmatrix} \geq 0.$$

Reversing the process in Subsection 2.5 whereby the above inequality is obtained, we deduce that a pair $(\boldsymbol{s}, \boldsymbol{g})$ belongs to the set Ω if and only if (a) $(\boldsymbol{s}, \boldsymbol{g})$ is an *admissible* pair of sales and generation, i.e., this pair is nonnegative and satisfies

$$g^f \le \mathrm{CAP}^f \quad \text{and} \quad \sum_{i=1}^{n} (s_{fi} - g_{fi}) = 0, \quad \forall\, f \in \mathcal{F},$$

and (b) there exists a pair of shadow prices (v, z) on the resources and transmission flows, respectively, such that the induced resource usage u and flow y are feasible, i.e., $d \le Du$ and $h \le Hy$, where

$$u \equiv \sum_{f \in \mathcal{F}} (B^f s^f + E^f g^f - \omega^f) + E^a a - \omega^a$$

and

$$y \equiv \sum_{f \in \mathcal{F}} (s^f - g^f) + a,$$

with

$$a \equiv Q^0 - S - \mathrm{Diag}(Q_i^0/P_i^0)\, p$$

and

$$p \equiv \boldsymbol{A}\,(Q^0 - e^0 - S) + [\, I - \boldsymbol{A}\mathrm{Diag}(Q_i^0/P_i^0)\,]\,[\,(DE^a)^T v + H^T z\,]$$

being the vectors of arbitrage quantities and nodal prices, respectively, which are induced by the tuple $(\boldsymbol{s}, \boldsymbol{g}, v, z)$. Based on this observation, we give below a sufficient condition on the model data for the set Ω to be nonempty, and hence for the qualitative results in the previous subsections to hold. Roughly speaking, this condition stipulates that there is an admissible pair of sales and generation by the firms such that the resources used and the induced transmission flows are feasible (conditions (b) and (c) below, respectively) and that e^0 is an optimal arbitrage amount when both the resource price and the transmission fee are equal to zero (see condition (a) and the proof below).

Proposition 5. Suppose that there exists an admissible pair $(\boldsymbol{s}, \boldsymbol{g})$ of sales and generation such that

(a) $P^0 - \mathrm{Diag}(P_i^0/Q_i^0)(e^0 + S) = G^T y$ for some y,

(b) $D \left[\sum_{f \in \mathcal{F}} (B^f s^f + E^f g^f - \omega^f) + E^a e^0 - \omega^a \right] \le d$, and

(c) $H \left[\sum_{f \in \mathcal{F}} (s^f - g^f) + e^0 \right] \le h$,

then $\Omega \neq \emptyset$.

Proof. It is not difficult to verify that the given pair $(\boldsymbol{s}, \boldsymbol{g})$ is an element of Ω by verifying the aforementioned necessary and sufficient condition for such membership, with $v \equiv 0$, $z \equiv 0$, $a \equiv e^0$, and $p \equiv \boldsymbol{A}(Q^0 - e^0 - S)$. Incidentally, condition (a) is equivalent to the condition that $a = e^0$ is an optimal solution to the arbitrage's problem with $\rho^* = 0$ and $w^* = 0$. □

4. A numerical example

An undergraduate student in the Department of Applied Mathematics & Statistics at the Johns Hopkins University, Grant Roch, wrote an AMPL code that implements a simple numerical example to illustrate the special model presented in Subsection 3.3. The PATH solver on the NEOS server at Argonne National Laboratory (http://www-neos.mcs.anl.gov/neos/) was used for solving the example, which is an endogenous-arbitrage problem with 2 firms and 4 regions. The ISO's constraint $Hy \le h$ is:

$$\frac{1}{3}\,y_2 + \frac{2}{3}\,y_3 \;\le\; h;$$

thus, $H = [\,0\ 1/3\ 2/3\ 0\,]$ and h is a scalar to be specified. This constraint corresponds to an upper bound on flow from node 1 to node 3 in a linearized DC load flow model in which nodes 1, 2, and 3 are arranged in triangular network with equal impedances for the three lines. Node 1 is assumed to be the hub node, while node 4 is assumed to be radially connected to node 1. There is only one resource; the resource allocator's constraint $Du \le d$ is a simple upper bound: $u \le d$; thus $D = 1$ and d is a scalar to be specified. The firms' generation cost functions are all linear: $c_{fi}(g_{fi}) \equiv c_{fi}\,g_{fi}$. Of course, these cost functions exclude the resource cost. Firms do not have pre-allocated resources, thus $\omega_f = 0$; firm f has only one resource usage constraint $E^f g^f = r^f$, which can be written as

$$\sum_{i=1}^{4} e_{fi}\, g_{fi} \;=\; r_f.$$

The remaining data of the problem are summarized in the following table. Results of our

Table 1. Data for numerical example

firm	node	c_{fi}	CAP_{fi}	e_{fi}	σ_f
1					0 or 1
1	1	50	50	.1	
1	2	0	0	0	
1	3	0	0	0	
1	4	10	50	.4	
2					0 or 1
2	1	0	0	0	
2	2	20	100	.3	
2	3	0	0	0	
2	4	0	0	0	

runs are reported in Table 2 below, which shows seven solutions in order to contrast the impact of various assumptions about the intensity of competition and the presence of resource and transmission constraints. The first four rows summarize the assumptions of each solution. In the energy market, the generation firms are assumed to hold one of the following conjectures: price taking (competitive, also called Bertrand), Cournot ($\beta_{fi} = 0$ MWh/($/MWh) in the fixed slope conjecture), or conjectured supply function ("CSF", $\beta_{fi} = 0.2$). In the resource market, firms are either price takers ($\sigma_f = 0$

Table 2. Results for numerical example

Case	Bertrand	Cournot I	Cournot II	CSF I	CSF II	CSF III	CSF IV
Energy market conjecture	Price taker	Cournot	Cournot	CSF	CSF	CSF	CSF
Resource price conjecture σ_f	0	0	1	0	1	1	1
Resource limit d	35	35	35	35	35	no limit	35
Transmission limit h	40	40	40	40	40	40	no limit
Total sales							
$f = 1$	50.0	43.3	45.4	42.9	46.2	52.9	69.1
$f = 2$	100.0	88.3	81.1	99.0	88.1	100.0	74.5
g_{11}	50.0	29.4	25.0	39.5	33.0	2.9	50.0
g_{14}	0	14	20	3	13.1	50.0	19.1
g_{22}	100.0	88.3	81.1	99.0	88.1	100.0	74.5
p_1	68.3	77.8	78.5	74.6	75.5	65.9	85.5
p_2	83.3	89.4	91.2	86.0	88.6	82.4	85.5
p_3	98.3	101.1	103.8	97.5	101.6	98.8	85.5
p_4	68.3	77.8	78.5	74.6	75.5	65.9	85.5
r_1	5.0	8.5	10.7	5.3	8.6	20.3	12.7
r_2	30.0	26.5	24.3	29.7	26.4	30.0	22.3
ρ^*	183.4	133.3	122.7	133.3	124.8	0.0	130.5
Power flow (1,3)	40.0	40.0	40.0	40.0	40.0	40.0	56.4
Profit, $f = 1$	0	626	800	484	634	2843	1565
Profit, $f = 2$	833	2601	2788	2578	2744	6238	1959
ISO profit	1800	1400	1523	1372	1570	1972	0
Net welfare	14858	14486	14298	14775	14518	16908	14864
Change in Welfare[a]	0	-373	-560	-83	-340	2049	6

[a]Compared to Bertrand case.

(\$/unit)/unit) or they conjecture that the resource price will rise if more is demanded ($\sigma_f = 1$). The results shown include the total sales (to all nodes) by each firm, generation from each plant, nodal prices, resource use by each producer, the resource prices, power flow in the constrained transmission link, producer and ISO profits, and total welfare. Total welfare is a standard measure of economic efficiency. It equals the sum of net benefits received by all parties, including consumer surplus (demand curve integral minus purchase costs), profits received by producers and the ISO, and revenue earned by the resource owner.

The first aspect of the obtained results we consider is the impact of different types of competition in the energy market, assuming producers are price takers in the resource market. Perfect competition yields the lowest consumer prices, equal to the marginal cost of supplying power to each node, including the shadow prices of transmission and the resource. By the famous welfare theorem, competition also yields the highest total welfare. The table indicates that CSF I prices are next lowest and Cournot I prices are highest, consistent with [11]. This is as expected, since a producer expects the least loss of market share in response to its raising price in the Cournot solution (where rivals are assumed not to respond to price increases) and the most loss in the competitive case (where it assumes that it loses its entire market share if it tries to charge more than the market price). Profits are highest when prices are highest (the Cournot solution). Meanwhile, the resource price is highest ($183.4/unit) under competition, because generators produce the most power in that case, which puts the most pressure on the resource. The high resource price means that producer $f = 1$ produces all its output from its plant at node $i = 1$, which has the lowest resource use rate of its two plants.

Second, we turn to the effect of a positive conjecture regarding the effect of resource demand upon resource price. This effect can be gauged by comparing Cournot solutions I and II, and, in the CSF case, CSF solutions I and II. Both comparisons show that such a conjecture lessens producer willingness to pay for the resource, resulting in a lower resource price. (For instance, ρ^* falls from 133.3 to 122.7 $/unit in the Cournot case.) Willingness to pay falls because when $\sigma_f > 0$, producer f becomes an oligopsonist with respect to the input resource, anticipating a marginal expenditure higher than the resource price. Meanwhile, this higher marginal resource expenditure increases the apparent marginal cost of production, ultimately translating into higher energy prices. Between the higher energy price and lower resource price, each producer earns $150–190 more profit than if they were price takers relative to the resource price. Welfare falls because artificially higher consumer prices mean that, on the margin, consumers are not buying power whose true social cost is less than its value to consumers.

Third, we consider the separate effects of relaxing the resource and transmission constraint. Under this particular set of assumptions, removing the resource constraint (resulting in no cost to the producer for using the resource) increases resource use from 35 to 50.3 (cf. solutions CSF II and III). Producer $f = 1$ switches nearly all of its production to its more resource-intensive plant (at node 4). Lower input costs then translate to significantly lower consumer prices for energy. In contrast, relaxing the transmission constraint from node 1 to node 3 has a less drastic effect on the equilibrium. Comparing solutions CSF II and IV, we see that flow on the link from $i = 1$ to $i = 3$ does increase from 40 to 56.4 MW, and firm $f = 1$ is now able to expand its production (and profit) at the expense of $f = 2$. Nodal energy prices are now the same everywhere, as they should be in the absence of binding transmission constraints. However, total sales expand only slightly (from 141.9 to 143.6 MW), and the effect on average prices (across nodes) is less than when the resource constraint is relaxed. Of course, the relative importance of the transmission and resource constraints depends on the assumptions made; the point of this example is that the input resource limit and producer conjectures about its price can significantly affect the solution.

5. Conclusion

Experience with restructured power markets indicates that the exercise of market power can often involve strategies much more sophisticated than the simple withholding of generation capacity. Power producers interact not only in power markets, but also in markets for resource inputs, such as emissions allowances and fuel. Evidence from California indicates that manipulation of natural gas and NO_x allowances markets may have allowed power producers to further increase their profits during the 2000–2001 crisis by providing justifications for increasing prices or shutting down generation capacity (e.g., [18]).

The models presented and analyzed in this paper represent a first step towards representing strategic interaction in both power and input markets while explicitly considering transmission network constraints. The models can represent Cournot competition in power markets, along with a more general form of competition we term “conjectured supply function” competition in which firms anticipate how rival sales will change if price changes. The conjectured supply function can be either of two types: a fixed-intercept/variable-slope version in which the intercept of the rival output-price relationship is predetermined, and a variable-intercept/fixed-slope version in which it is the slope of the function that is preset. The former version yields a nonlinear complementarity problem, while the latter instead gives an LCP. Meanwhile, in input markets, either Bertrand (price-taking) or “conjectured resource price function” behavior is modeled. In the latter case, firms anticipate that if they change their resource consumption from the equilibrium value, the price of the resource will rise. Finally, power producers’ expectations concerning arbitrage among nodes in the power market is represented in two ways. One approach has the producers anticipating how arbitragers will react if prices change (“endogenous arbitrage”), while in the other, producers assume that the amount of arbitrage is exogenous.

Existence and uniqueness properties are proven for several of these models, including the variable-intercept and, under specified conditions, variable-slope versions. Equivalence of the endogenous and exogenous arbitrage solutions is shown under certain conditions, permitting use of whichever formulation is computationally most convenient. In future work, we will apply these models to actual power markets.

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