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Nash-Cournot Equilibria in Electric Power Markets with Piecewise Linear Demand Functions and Joint Constraints

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Most previous Nash-Cournot models of competition among electricity generators have assumed smooth demand (price) functions, facilitating computation and proofs of existence and uniqueness. However, nonsmooth demand functions are an important feature of real power markets due, for example, to price caps and generator recognition of transmission constraints that limit exports. A more general model of Nash-Cournot competition on networks is proposed that accounts for these features by including (1) concave piecewise-linear demand curves and (2) joint constraints that include variables from other generating companies within the profit maximization problems for individual generators. The piecewise demand curves imply, in general, a nonmonotone multivalued variational inequality problem. Thus, for instance, imposition of a price cap can destroy the uniqueness properties found in previous models, so that distinct solutions can yield different sets of profits for market participants. The joint constraints turn the equilibrium problem into a quasi-variational inequality, which also can yield multiple solutions. The formulation poses computational challenges that can cause Lemke's algorithm to fail; a restricted formulation is proposed that can be solved by that algorithm.

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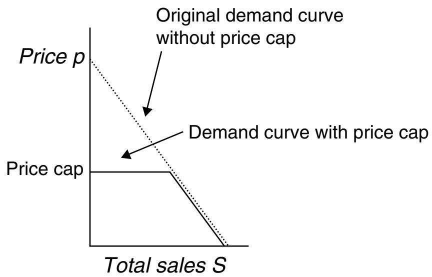
1. Introduction

One of the major areas of application of complementarity-based models of economic equilibria is electric power markets; see, e.g., Amundsen et al. (2001), Bushnell (2003), Daxhalet and Smeers (2001), Day et al. (2002), Hobbs (2001), Hobbs and Helman (2004), Hobbs and Pang (2004), Rivier et al. (2001), Wei and Smeers (1999), and Yao et al. (2005). One reason is that this is an economically crucial industry that has been undergoing a transition from tight regulation to competition subject to loose regulatory constraints. Furthermore, this transition has on occasion resulted in spectacular failures, such as the California 2000–2001 crisis. Another reason is that technology and cost information is widely available for the power industry, which facilitates modeling, while at the same time the unique characteristics of electricity transmission, such as Kirchhoff's laws, present intriguing challenges to the modeler.

However, there are important features of power markets that such models have omitted. One is price caps, which exist in all U.S. markets. For instance, the U.S. Federal Energy Regulatory Commission imposed a \$250/megawatt-hour (MWh) cap on market prices in the western United States in response to the California crisis. Such caps mean that the demand (price) functions faced by sellers of power have a horizontal segment (Figure 1), and that the linear or otherwise smooth demand functions assumed by most complementarity-based models are inappropriate, unless prices never approach the caps. In general, demand functions might not be smooth, but most existing complementarity-based models have not accommodated such functions. An exception is Yao et al. (2005), where the authors are able to solve a price-capped market model that makes two restrictive assumptions, thereby avoiding the difficulties we encounter below of multiple equilibria and the potential failure of Lemke's algorithm. These assumptions are that only one firm owns generation capacity at each network node, and that a generator at a node does not consider how changes in its output might affect prices elsewhere in the network. These assumptions mean that other firms' decision variables do not appear in each firm's profit maximization problem. In this paper, we do not impose such restrictive assumptions; thus, our treatment is much broader and more realistic.

Another important feature omitted in most complementarity-based models is the possibility that an electricity producer can recognize joint constraints that involve its primal

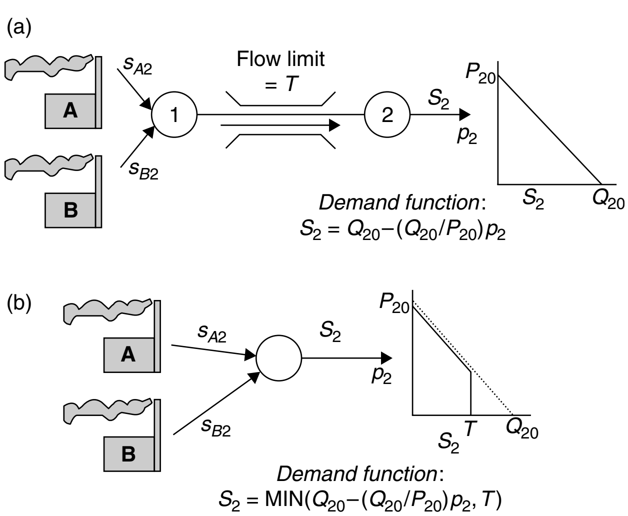
Figure 1. A price function with cap.



decision variables along with primal variables controlled by other generators. For instance, regulators might impose an upper bound on the market share of the few largest producers in some markets, or on the proportion of transmission capacity that is sold to such producers, as is the case for transmission capacity into the Netherlands. Perhaps the most important example would be a recognition by a generator that its sales and generation are limited by available transmission capacity, less that capacity which is already taken up by sales and generation by other producers. For instance, consider Figure 2(a), where there are two generation companies A and B at node 1, and consumers having an affine demand function at node 2. Sales by company f to node 2 are designated s_{f2}, and the total amount sold to node 2 equals S_2. A transmission line with capacity T links the two nodes and constrains sales so that $S_2 \leqslant T$. If T is smaller than the quantity intercept Q_{20} of the demand curve, then the effective demand curve facing the two producers is piecewise linear, as shown in Figure 2(b), with an upper bound equal to the total quantity sold. More complex sets of constraints on sales and generation result from more elaborate network topologies.

As Oren (1997) and Stoft (1999) have shown, if Cournot producers explicitly include such transmission limits in their constraint sets, the resulting equilibrium problem is

Figure 2. A demand function with transmission cap.



a quasi-variational inequality with possibly multiple solutions. For instance, producer A might use up all the capacity, leaving none for B, while another equilibrium might be an equal splitting of that capacity. Wei and Smeers (1999) point out that the problem of multiple equilibria can be eliminated by assuming that the joint constraints are instead incorporated in the producers' objective function using Lagrangian multipliers (prices), and that the prices, and thus marginal valuations of the constraint, are the same for all producers. This result is exploited in most complementarity-based models of transmission-constrained Cournot competition, but it has the conceptual drawback of being equivalent to assuming that producers are naïve price takers with respect to transmission (Bertrand assumption), believing that they cannot affect the price of transmission by their actions. (This approach has computational advantages, however, as pointed out in §4.1.)

A more realistic, but computationally challenging method for including transmission limits in producer models is to embed the first-order conditions for a transmission system operator who sets prices for its constraints to clear the market for transmission services (Cardell et al. 1997, Ehrenmann 2004, Hu et al. 2004). These conditions are described in §3.1. The producers then anticipate how their outputs and sales and those by other producers affect the amount of transmission capacity that is utilized and the resulting transmission prices. The resulting formulation of the producer's profit maximization problem is a mathematical program with equilibrium constraints (MPEC), and the calculation of an equilibrium among such MPECs is an equilibrium problem with equilibrium constraints (EPEC) (Daxhalet and Smeers 2001). In general, solutions might not exist for this EPEC, or there might be multiple solutions (Daxhalet and Smeers 2001, Hobbs and Helman 2004); most importantly, the state of the art for computing a solution to an EPEC is in its infancy at best. This is an important topic that requires further research.

In this paper, we generalize complementarity-based models of Nash-Cournot oligopolistic electric power markets to include concave demand functions that are piecewise linear. Price-cap-constrained demand functions are a special case of such functions. These models also include linear joint constraints within generator profit maximization problems. The resulting generalization is a significant departure from the models studied previously in Metzler et al. (2003), Hobbs and Pang (2004), and Pang et al. (2003), where the price functions are linear. The new model features are computationally challenging; for one thing, they immediately invalidate the solution methods employed for the previous models, which rely on a straightforward variational inequality/complementarity formulation that needs to be extended to accommodate the nondifferentiable objective functions in the firms' profit-maximization problems. In this paper, we begin with a *multivalued* complementarity formulation of the equilibrium problem, from which an equivalent single-valued linear complementarity problem

(LCP) formulation is derived. We then provide examples to illustrate some special features of both complementarity formulations and propose a restricted formulation that can be successfully solved by the well-known Lemke algorithm (Cottle et al. 1992, Lemke 1965).

The class of piecewise linear price functions provides an effective modeling device to handle price caps. The introduction of joint constraints effectively turns the equilibrium problem into a quasi-variational inequality (QVI) (Harker 1991, Pang and Fukushima 2005), as opposed to a familiar variational inequality (Facchinei and Pang 2003), which is the common framework for a vanilla Nash-Cournot model. The "quasi-ness" of this variational problem is a feature that requires special attention; the restricted LCP developed in §4.1 is a targeted proposal to deal with such a QVI.

We begin our technical discussion with a basic model of oligopolistic electric power markets with multiple generators and an independent system operator (ISO), who is the principal agent for electricity transmission. Although our previous models have considered additional players in the markets (such as an allocator of inputs to production and an arbitrager) and refinements of the basic model (such as price function conjectures and endogenous arbitrage), cf. Metzler et al. (2003), Hobbs and Pang (2004), and Pang et al. (2003), the piecewise feature of the price functions and the sales caps add two novel dimensions to the model that have not previously been treated. For this reason, to simplify the analysis while not sacrificing its generality, we will treat the basic model exclusively and will discuss its extensions only minimally.

2. The Firms' Problems

Each generator firm is labelled by a letter f, which is an element of the finite set $\mathcal{F}$. The firms' profits are revenues less costs; in turn, the revenues are equal to regional sales times the corresponding nodal prices, and the costs are generation costs and transmissions fees, the latter paid to the ISO. The constraints are simple quantity balances between sales and generation, the latter subject to capacity limits. Specifically, firm f's optimization problem is as follows (cf. the model in Metzler et al. 2003): with the transmission fee w_i and the rival firms' sales $s_{-fi} \equiv \{s_{hi}\colon h \neq f\}$, taken as exogenous to this optimization problem and yet endogenous to the overall equilibrium model, firm f computes its nodal sales s_{fi} and generations g_{fi} for all $i \in \mathcal{N}$, which is the set of nodes in the electricity network, to

$$\begin{aligned}
&\text{maximize} \quad \sum_{i\in\mathcal{N}} [s_{fi}p_i(S_i) - c_{fi}g_{fi} - (s_{fi} - g_{fi})w_i] \\
&\text{subject to} \quad \sum_{i\in\mathcal{N}} s_{fi} = \sum_{i\in\mathcal{N}} g_{fi}, \quad \text{and} \\
&\qquad \left\{ \begin{array}{c} 0 \leqslant s_{fi}, \\ 0 \leqslant g_{fi} \leqslant \mathrm{CAP}_{fi}, \\ S_i \equiv \sum_{h\in\mathcal{F}} s_{hi} \leqslant \sigma_i, \end{array} \right\} \quad \forall\, i \in \mathcal{N}.
\end{aligned} \tag{1}$$

Here c_{fi}, CAP_{fi}, and σ_i are positive model constants, denoting unit generation costs, generation capacities, and regional sales caps, respectively. The fee w_i is interpreted as the price paid to move power from an arbitrary hub node (taken to be one; see below) to node i ($\neq 1$); it is a variable from the point of view of the market, but it is exogenous to the firm. Instead of the sales cap constraint

$$\sigma_i - \sum_{h\in\mathcal{F}} s_{hi} \geqslant 0, \tag{2}$$

more general constraints of the kind

$$\sum_{i\in\mathcal{N}} \delta^s_{ji} \sum_{h\in\mathcal{F}} s_{hi} + \sum_{i\in\mathcal{N}} \delta^g_{ji} \sum_{h\in\mathcal{F}} g_{hi} \leqslant \zeta_j,$$

where δ^s_{ji}, δ^g_{ji}, and ζ_j are constant coefficients, with the index j suggesting the possibility of several of these constraints, that involve all firms' sales and generation can be treated. For simplicity, we focus on (2).

The above model of the generation firms' decision making is based on a bilateral market model in which generators contract with consumers to deliver electricity, with generators paying the cost of transmission from the point of generation to the point of consumption. Generators then provide a schedule of injections and withdrawals to the ISO, who charges the generator for the use of the grid. The ISO may in addition operate a spot market in which generators can unilaterally sell power and consumers can unilaterally buy power at spot prices that are defined on a nodal basis. This mix of bilateral and spot markets is the model endorsed by the U.S. Federal Energy Regulatory Commission and followed by most U.S. ISOs in operating day-ahead markets. In fact, the bulk of transactions take place on a bilateral basis, but a significant amount flows through the spot market. (For simplicity, we model the market as entirely bilateral, but the incorporation of a spot market is a straightforward extension, as we explain in §3.1.) It is not physically possible to unambiguously apportion responsibility for power flows to individual generators and consumers. However, the metering, accounting, and settlement systems used by most U.S. ISOs allow financial responsibility to be assigned to generators in the manner we model above, while simultaneously ensuring that the physical flows implied by the transactions are feasible (which we model in the manner described in §§3.1 and 3.2). An important characteristic of transmission pricing as practiced in the United States is that the fee for transferring power from i to j is the negative of the fee for moving power in the opposite direction. Furthermore, superposition applies: the fee from i to j plus the fee from j to k equals the fee from i to k. These characteristics result from the ISO using scarcity pricing to determine fees (Schweppe et al. 1988); as the ISO and market-clearing models in §§3.1 and 3.2 imply, fees are set to clear the market for scarce transmission capacity. Thus, for instance, if a generator schedules a power flow from i to j that would worsen congestion on a binding transmission constraint, then the generator pays the opportunity cost of that constraint. In contrast, a flow from j to i would relieve that congestion, and therefore a generator scheduling such a flow receives a payment equal to that opportunity cost. These characteristics allow us to express the transmission fees paid by the generator in the manner shown in the objective function of (1). The first characteristic means that power flowing from a generator to the hub node is paid w_i because w_i is the fee for flow in the opposite direction. The second characteristic (superposition) means that sales from a generator node to a consuming node can be modelled as being routed through an arbitrary hub node.

Defined on the entire real line, the price function $p_i(S_i)$ is postulated to be a concave, strictly decreasing, piecewise linear function of the regional sales S_i. More general assumptions could be made in which demand functions at different i are interdependent, e.g., because of zonally averaged pricing as is presently proposed in the California market redesign. However, we assume zero cross-price elasticities across space and time. Specifically, letting $\alpha_{i1} < \alpha_{i2} < \cdots < \alpha_{im}$ be the breakpoints of the function $p_i(S_i)$, we write the price function as follows:

$$p_i(\tau) \equiv \begin{cases} P_{i0} - \beta_{i0}\tau & \text{if } -\infty < \tau \leqslant \alpha_{i1}, \\ p_i(\alpha_{ij}) - \beta_{ij}(\tau - \alpha_{ij}) & \\ \quad \text{if } \alpha_{ij} \leqslant \tau \leqslant \alpha_{ij+1},\ 1 \leqslant j \leqslant m-1, & \\ p_i(\alpha_{im}) - \beta_{im}(\tau - \alpha_{im}) & \text{if } \alpha_{im} \leqslant \tau < \infty, \end{cases} \tag{3}$$

where $\beta_{im} > \cdots > \beta_{i0} \geqslant 0$ are the negatives of the slopes of $p_i(\tau)$ in the respective intervals of linearity. Because $p_i(\tau)$ is postulated to be a concave function defined on the whole real line, it follows that $p_i(\tau)$ is continuous; the above representation confirms this continuity. Furthermore, because the regional sales S_i is restricted to be nonnegative, we may assume without loss of generality that the first breakpoint α_{i1} is a positive scalar and that the function $p_i(\tau)$ for $\tau < 0$ extends linearly (to the left) from $\tau = 0$. Note that a function of the form

$$p_i(S_i) = \min\left(P_i^{\text{CAP}}, P_{i0} - \frac{P_{i0}}{Q_{i0}} S_i\right)$$

(where P_i^{CAP} is a positive constant), which models price caps, is trivially a piecewise linear function. In contrast, the explicit sales cap $S_i \leqslant \sigma_i$ or its generalization can be used to represent transmission limits into a region.

It is easy to write the subgradients $\partial p_i(\tau)$ of the concave function $p_i(\tau)$ for $\tau \geqslant 0$:

$$p_i(\tau) = \begin{cases} \{-\beta_{i0}\} & \text{if } \tau = \alpha_{i0}, \\ \{-\beta_{ij}\} & \text{if } \tau \in (\alpha_{ij}, \alpha_{ij+1}), \\ [-\beta_{ij+1}, -\beta_{ij}] & \text{if } \tau = \alpha_{ij+1}, \end{cases} \tag{4}$$

where we have let $\alpha_{i0} = 0$ and $\alpha_{im+1} = \infty$. The next lemma shows that firm f's nodal revenue function $s_{fi} \mapsto s_{fi} p_i(s_{fi} + S_{-fi})$, where $S_{-fi} \equiv \sum_{f \neq h \in \mathcal{F}} s_{hi}$, is concave in s_{fi} for fixed S_{-fi}; the lemma also gives a natural expression for the subgradients of this function.

Lemma 1. *If $p(\tau)$ is a nonincreasing concave function on $\Re$, then, for any constant c, the function $r(\tau) \equiv \tau p(\tau + c)$ is concave for $\tau \geqslant 0$; moreover, $\partial r(\tau) = p(\tau + c) + \tau \partial p(\tau + c)$ for all $\tau \geqslant 0$.*

Proof. For $\theta \in [0, 1]$, we have, for any nonnegative τ and τ',

$$\begin{aligned} &r(\theta\tau + (1-\theta)\tau') - \theta r(\tau) - (1-\theta) r(\tau') \\ &\quad = [\theta\tau + (1-\theta)\tau'] p(\theta(\tau + c) + (1-\theta)(\tau' + c)) \\ &\qquad - \theta\tau p(\tau + c) - (1-\theta)\tau' p(\tau' + c) \\ &\quad \geqslant [\theta\tau + (1-\theta)\tau'][\theta p(\tau + c) + (1-\theta) p(\tau' + c)] \\ &\qquad - \theta\tau p(\tau + c) - (1-\theta)\tau' p(\tau' + c) \\ &\quad = -\theta(1-\theta)(\tau - \tau')(p(\tau + c) - p(\tau' + c)) \geqslant 0, \end{aligned}$$

where the last inequality follows from the nonincreasing property of p. To prove the subgradient formula of $\partial p(\tau)$, we note that, for any nonnegative τ and τ' and any $a \in \partial p(\tau' + c)$,

$$\begin{aligned} &\tau p(\tau + c) - \tau' p(\tau' + c) \\ &\quad = (\tau - \tau') p(\tau' + c) + (\tau - \tau')(p(\tau + c) - p(\tau' + c)) \\ &\qquad + \tau'(p(\tau + c) - p(\tau' + c)) \\ &\quad \leqslant (\tau - \tau')(p(\tau' + c) + a\tau'), \end{aligned}$$

where the last inequality holds by the definition of a and because p is nonincreasing. Hence, $p(\tau' + c) + \tau' \partial p(\tau' + c) \subseteq \partial r(\tau')$. The reverse inclusion follows from the well-known product rule of subdifferentials of convex functions (Clarke 1983, Proposition 2.3.13). (It is easy to show that the function r is actually Fréchet differentiable at $\tau = 0$ with the derivative $r'(0) = p(c)$.) □

2.1. Nonmonotone Multivalued QVI Formulation

With s_{-fi} and w_i fixed, (1) is a concave maximization problem; as such, its first-order conditions are both necessary and sufficient for optimality. Because the objective function is nondifferentiable, these conditions can be stated in terms of a multivalued variational inequality (VI). Specifically, let

$$X_f(s^{-f}) \equiv \{(s^f, g^f) \text{ satisfies the constraints in (1)}\}$$

be the feasible set of (1) that depends on rival firms' sales S_{-fi}, where the dependence is due to the sales cap constraint (2). It follows that (s^f, g^f) is optimal for (1) if and only if $(s^f, g^f) \in X_f(s^{-f})$ and there exist $b_{fi} \in \partial p_i(S_i)$ for all $i \in \mathcal{N}$ such that

$$\sum_{i\in\mathcal{N}}[(\hat{s}_{fi}-s_{fi})(-p_i(S_i)-s_{fi}b_{fi}+w_i^*) + (c_{fi}-w_i^*)(\hat{g}_{fi}-g_{fi})] \geqslant 0$$

for all $(\hat{s}^f, \hat{g}^f) \in X_f(s^{-f})$. The complexity of this problem is due not only to the fact that the set $X_f(s^{-f})$ depends on the rival firms' sales (therefore resulting in a *quasi-variational inequality*), but also to the fact that the set-valued map Φ from $\Re^{|\mathcal{F}\|\mathcal{N}|}$ into closed-convex subsets of $\Re^{|\mathcal{F}\|\mathcal{N}|}$ defined by

$$\Phi(s) \equiv (-p_i(S_i) - s_{fi}\partial p_i(S_i))_{(f,i)\in\mathcal{F}\times\mathcal{N}} \quad \text{for } s=(s_{fi})_{(f,i)\in\mathcal{F}\times\mathcal{N}}$$

is not monotone, as illustrated by the example below. Recall that Φ is monotone on $\Re_+^{|\mathcal{F}\|\mathcal{N}|}$ if and only if $(\tilde{s}-\hat{s})^T(\tilde{u}-\hat{u}) \geqslant 0$ for all $\tilde{s}$, $\hat{s} \geqslant 0$ and $\tilde{u} \in \Phi(\tilde{s})$ and $\hat{u} \in \Phi(\hat{u})$.

Example 1. Let $\mathcal{N} = \{1\}$ and $\mathcal{F} = \{1, 2\}$, and consider the piecewise linear function: for a constant $c \in (0, 1)$,

$$p(S) \equiv \begin{cases} 1+c-cS & \text{if } S \leqslant 1, \\ 1-c^{-1}(S-1) & \text{if } S \geqslant 1. \end{cases}$$

We have

$$\partial p(S) \equiv \begin{cases} \{-c\} & \text{if } S < 1, \\ [-1/c, -c] & \text{if } S = 1, \\ \{-1/c\} & \text{if } S > 1. \end{cases}$$

Hence,

$$\Phi(s_1, s_2) = \begin{cases} \left\{\begin{pmatrix} -1-c+c(2s_1+s_2) \\ -1-c+c(s_1+2s_2)\end{pmatrix}\right\} & \text{if } s_1+s_2<1, \\ (-1-s_1[-1/c,-c]) \times (-1-s_2[-1/c,-c]) & \text{if } s_1+s_2=1, \\ \left\{\begin{pmatrix} -1-c^{-1}+c^{-1}(2s_1+s_2) \\ -1-c^{-1}+c^{-1}(s_1+2s_2)\end{pmatrix}\right\} & \text{if } s_1+s_2>1. \end{cases}$$

Consider the two pairs of vectors $(\tilde{s}_1, \tilde{s}_2) \equiv (0.45, 0.55)$ and $(\hat{s}_1, \hat{s}_2) \equiv (0.55, 0.45)$. We then have

$$\Phi(\tilde{s}_1, \tilde{s}_2) = (-1-0.45[-1/c,-c]) \times (-1-0.55[-1/c,-c])$$

and

$$\Phi(\hat{s}_1, \hat{s}_2) = (-1-0.55[-1/c,-c]) \times (-1-0.45[-1/c,-c]).$$

Choose $(\tilde{u}_1, \tilde{u}_2) = (0.45/c, 0.55c)$ and $(\hat{u}_1, \hat{u}_2) = (0.55c, 0.45/c)$. We then have

$$\begin{pmatrix} \tilde{s}_1-\tilde{s}_2 \\ \hat{s}_1-\hat{s}_2 \end{pmatrix}^T \begin{pmatrix} \tilde{u}_1-\tilde{u}_2 \\ \hat{u}_2-\hat{u}_2 \end{pmatrix} = -0.1(0.45/c-0.55c) < 0$$

for $0 < c < \sqrt{0.45/0.55}$, establishing that Φ is not monotone.

The nonmonotonicity of Φ means that the extensive theory and methods for monotone multivalued VIs, as championed by the work of Auslender and Teboulle (see Chapter 12 in Facchinei and Pang 2003), are not applicable to our equilibrium problem on hand. Instead, we develop a restricted, single-valued, linear complementarity formulation for this problem and establish its solution by Lemke's method (Cottle et al. 1992, Lemke 1965). Another implication of the lack of monotonicity is that uniqueness of a solution to the equilibrium problem is no longer ensured; in fact, the numerical example given above can easily be turned into one where both vectors $\tilde{s}$ and $\hat{s}$ are solutions. See also Example 2 below and the numerical example in §5, where multiple equilibria exist.

2.2. Complementarity Formulation of Piecewise Quadratic Revenue

The key to the single-valued formulation of the overall equilibrium problem is to cast the piecewise linearity of the price functions in terms of certain complementarity conditions. For this purpose, we let

$$r_{fi}(s_{fi}, S_{-fi}) \equiv s_{fi}p_i(s_{fi}+S_{-fi})$$

be firm f's sales revenue, which is a *piecewise quadratic function* of the firm's variables s_{fi}. The key to the reformulation lies in the expression of the elements of the subdifferential $\partial_{s_{fi}} r_{fi}(s_{fi}, S_{-fi})$, which are the marginal sales revenues, in terms of some special complementarity relations. Let

$$\alpha'_{ij} \equiv \alpha_{ij} - \alpha_{ij-1} \quad \forall\, j=1,\ldots,m+1$$

be the lengths of the respective intervals of linearity of $p_i(\tau)$ for $\tau \geqslant 0$. Let $\beta'_{i0} \equiv \beta_{i0}$ and define inductively,

$$\beta'_{ij} \equiv \beta_{ij} - \beta_{ij-1} \quad \forall\, j=1,\ldots,m.$$

Clearly, $\beta_{ij} = \sum_{k=0}^{j}\beta'_{ik}$. We have the following result, whose proof consists essentially of verifying the identified complementarity description of the subdifferential $\partial p_i(S_i)$ in (4).

Proposition 1. *For nonnegative s_{fi} and S_i, a scalar a belongs to $\partial_{s_{fi}} r_{fi}(s_{fi}, S_{-fi})$ if and only if there exist scalars $\{\tau_{fi1},\ldots,\tau_{fim}\}$ and $\{v_{fi1},\ldots,v_{fim}\}$ such that*

$$a = P_{i0} - \beta_{i0}\left(s_{fi}+S_i-\sum_{j=1}^{m}\tau_{fij}\right) - \sum_{j=1}^{m}\beta_{ij-1}\tau_{fij} - \sum_{j=1}^{m}\beta'_{ij}v_{fij} \quad (5)$$

and

$$0 \leqslant v_{fi1} \perp \alpha'_{i1} - \left(S_i - \sum_{j=1}^{m} \tau_{fij}\right) \geqslant 0,$$
$$0 \leqslant v_{fij+1} \perp \alpha'_{ij+1} - \tau_{fij} \geqslant 0 \quad \forall\, j = 1, \ldots, m-1, \tag{6}$$
$$0 \leqslant \tau_{fij} \perp s_{fi} - v_{fij} + \tau_{fij} \quad \forall\, j = 1, \ldots, m.$$

Proof. First write

$$S_i = \sum_{j=0}^{m} \tau_{fij}, \tag{7}$$

where τ_{fij} denotes the portion of S_i in the interval $[\alpha_{ij}, \alpha_{ij+1}]$ (recall that $\alpha_{i0} = 0$ and $\alpha_{im+1} = \infty$), satisfying

$$0 \leqslant \tau_{fij} \leqslant \alpha'_{ij+1} \quad \forall\, j = 0, \ldots, m,$$

and

$$(\alpha'_{ij+1} - \tau_{fij})\tau_{fij+1} = 0 \quad \forall\, j = 0, \ldots, m-1.$$

The subscript f in τ_{fij} indicates that this variable is used in firm f's problem to represent segment j of the piecewise linear demand curve at region i. Note that $\tau_{fij} < \alpha'_{ij+1} \Rightarrow \tau_{fik} = 0$ for all $k > j$. In terms of the above representation, we have

$$p_i(S_i) = P_{i0} - \sum_{j=0}^{m} \beta_{ij} \tau_{fij}.$$

From (7), we have

$$\tau_{fi0} \equiv S_i - \sum_{j=1}^{m} \tau_{fij}. \tag{8}$$

Suppose that $a \in \partial_{s_{fi}} r_i(s_{fi}, s_{-fi})$. Then,

$$a = p_i(S_i) + s_{fi} b = P_{i0} - \sum_{j=0}^{m} \beta_{ij} \tau_{fij} + s_{fi} b$$

for some $b \in \partial p_i(S_i)$. If $j \in \{0, 1, \ldots, m\}$ is the first index such that $\tau_{fij} < \alpha'_{ij+1}$, then defining

$$v_{fik} \begin{cases} \equiv s_{fi} + \tau_{fik} & \text{for } k = 1, \ldots, j-1, \\ \equiv s_{fi} + \tau_{fij} & \text{for } k = j \text{ and } \tau_{fij} > 0, \\ \in [0, s_{fi}] & \text{for } k = j \text{ and } \tau_{fij} = 0, \\ \equiv 0 & \text{for } k = j+1, \ldots, m, \end{cases}$$

we see that the complementarity conditions (6) are immediately satisfied. We verify that (5) also holds. Indeed, if $\tau_{fij} > 0$, then $S_i \in (\alpha_{ij}, \alpha_{ij+1})$ and $b = -\beta_{ij}$. Hence,

$$\begin{aligned} a &= P_{i0} - \sum_{k=0}^{m} \beta_{ik} \tau_{fik} - s_{fi} \beta_{ij} \\ &= P_{i0} - \beta_{i0} \tau_{fi0} - \sum_{k=1}^{m} \beta_{ik} \tau_{fik} - \sum_{k=0}^{j} \beta'_{ik} s_{fi} \\ &= P_{i0} - \beta_{i0}\left(s_{fi} + S_i - \sum_{k=1}^{m} \tau_{fik}\right) - \sum_{k=1}^{m} \beta_{ik} \tau_{fik} \\ &\quad - \sum_{k=1}^{j} \beta'_{ik}(v_{fik} - \tau_{fik}) \\ &= P_{i0} - \beta_{i0}(s_{fi} + S_i) - \sum_{k=1}^{m} (\beta_{ik} - \beta'_{ik} - \beta_{i0}) \tau_{fik} - \sum_{k=1}^{m} \beta'_{ik} v_{fik} \\ &= P_{i0} - \beta_{i0}(s_{fi} + S_i) - \sum_{k=1}^{m} (\beta_{ik-1} - \beta_{i0}) \tau_{fik} - \sum_{k=1}^{m} \beta'_{ik} v_{fik}, \end{aligned}$$

which is (5). If $\tau_{fij} = 0$, then $S_i = \alpha_{ij}$, which yields $b = -\beta_{i0}$ if $j = 0$ or $b \in [-\beta_{ij}, -\beta_{ij-1}]$ if $j > 0$. In the former case, the same derivation as above establishes (5). If $j > 0$, write $b = -\beta_{ij} + \zeta(-\beta_{ij-1} + \beta_{ij})$ for some $\zeta \in [0, 1]$. Let $v_{fij} \equiv (1 - \zeta) s_{fi}$. Then, $v_{fij} \in [0, s_{fi}]$. Moreover,

$$\begin{aligned} a &= P_{i0} - \sum_{k=0}^{m} \beta_{ik} \tau_{fik} + s_{fi} b \\ &= P_{i0} - \sum_{k=0}^{m} \beta_{ik} \tau_{fik} - s_i \beta_{ij} + \zeta s_{fi} \beta'_{ij} \\ &= P_{i0} - \beta_{i0} \tau_{fi0} - \sum_{k=1}^{m} \beta_{ik} \tau_{fik} - \sum_{k=0}^{j} \beta'_{ik} s_{fi} + (s_{fi} - v_{fij}) \beta'_{ij}. \end{aligned}$$

At this point, the above proof applies. This completes the proof of the "only if" part. To prove the converse, suppose that $\{\tau_{fi1}, \ldots, \tau_{fim}\}$ and $\{v_{fi1}, \ldots, v_{fim}\}$ exist, satisfying (5) and (6). Define τ_{fi0} by (8). We must have $\tau_{fi0} \geqslant 0$. Indeed, if $\tau_{fi0} < 0$, then $v_{fi1} = 0$, which then implies $\tau_{fi1} = 0$. Hence, $\tau_{fik} = 0$ for all $k > 1$. Therefore, $\tau_{fi0} = S_i \geqslant 0$, which is a contradiction. Consequently, $\tau_{fi0} \geqslant 0$, as claimed. Furthermore, we must have $\tau_{fij+1}(\alpha_{ij+1} - \tau_{fij}) = 0$ for all $j = 0, 1, \ldots, m-1$. Indeed, suppose that $\tau_{fij} < \alpha_{ij+1}$. We must have $v_{fij+1} = 0$. Hence, $0 \leqslant \tau_{fij+1} \perp s_{fi} + \tau_{fij+1} \geqslant 0$, which implies $\tau_{fij+1} = 0$. From here on, we can reverse the argument in the proof of the "only if" statement. □

3. Linear Complementarity Formulation of the Nash Model

Before giving the optimality conditions for firm f's optimization problem (1), we first eliminate one of the generation variables, thereby converting the sales-generation equality constraint into an inequality. Specifically, using the equation

$$g_{f1} \equiv \sum_{i \in \mathcal{N}} s_{fi} - \sum_{1 \neq i \in \mathcal{N}} g_{fi}, \tag{9}$$

we can reformulate the optimization problem (1) equivalently as a concave maximization problem in firm f's sales variables $\{s_{fi}: i \in \mathcal{N}\}$ and generation variables $\{g_{fi}: i \in \mathcal{N} \setminus \{1\}\}$ parameterized by rival firms' sales variables $\{s_{-fi}: i \in \mathcal{N}\}$:

$$\text{maximize} \quad \sum_{i \in \mathcal{N}} [s_{fi}(p_i(S_i) - c_{f1}) - (w_i - w_1) s_{fi}] - \sum_{1 \neq i \in \mathcal{N}} [(c_{fi} - c_{f1}) - (w_i - w_1)] g_{fi},$$

$$\text{subject to} \quad \sum_{1 \neq i \in \mathcal{N}} g_{fi} - \sum_{i \in \mathcal{N}} s_{fi} \leqslant 0,$$

$$\sum_{i\in\mathcal{N}} s_{fi} - \sum_{1\neq i\in\mathcal{N}} g_{fi} \leqslant \mathrm{CAP}_{f1},$$

$$\left\{\begin{array}{c} 0\leqslant s_{fi}, \\ S_i \equiv \sum_{h\in\mathcal{F}} s_{hi} \leqslant \sigma_i, \end{array}\right\} \quad \forall\, i\in\mathcal{N}, \quad \text{and}$$

$$0\leqslant g_{fi} \leqslant \mathrm{CAP}_i \quad \forall\, i\in\mathcal{N}\backslash\{1\}.$$

By Proposition 1, it follows that for fixed but arbitrary $s_{-fi}\geqslant 0$, (g^f, s^f) is an optimal solution of the above optimization problem if and only if there exist scalars $\{\tau_{fi1},\ldots,\tau_{fim}\}$ and $\{v_{fi1},\ldots,v_{fim}\}$ and multipliers ψ_f, γ_{fi}, and φ_{fi} such that

$$0\leqslant s_{fi} \perp c_{f1} - P_{i0} + \beta_{i0}\left(s_{fi}+S_i-\sum_{j=1}^{m}\tau_{fij}\right)+\sum_{j=1}^{m}\beta_{ij-1}\tau_{fij} + \sum_{j=1}^{m}\beta'_{ij}v_{fij}+w_i-w_1-\psi_f+\gamma_{f1}+\varphi_{fi}\geqslant 0 \quad \forall\, i\in\mathcal{N},$$

$$0\leqslant g_{fi}\perp (c_{fi}-c_{f1})-(w_i-w_1)+\psi_f-\gamma_{f1}+\gamma_{fi}\geqslant 0 \quad \forall\, i\in\mathcal{N}\backslash\{1\},$$

$$0\leqslant \psi_f \perp \sum_{i\in\mathcal{N}} s_{fi} - \sum_{1\neq i\in\mathcal{N}} g_{fi}\geqslant 0,$$

$$0\leqslant \gamma_{f1}\perp \mathrm{CAP}_{f1}-\sum_{i\in\mathcal{N}} s_{fi}+\sum_{1\neq i\in\mathcal{N}} g_{fi}\geqslant 0,$$

$$0\leqslant \gamma_{fi}\perp \mathrm{CAP}_{fi}-g_{fi}\geqslant 0 \quad \forall\, i\in\mathcal{N}\backslash\{1\},$$

$$0\leqslant \varphi_{fi}\perp \sigma_i - S_i\geqslant 0 \quad \forall\, i\in\mathcal{N},$$

$$0\leqslant v_{fi1}\perp \alpha'_{i1}-\left(S_i-\sum_{j=1}^{m}\tau_{fij}\right)\geqslant 0 \quad \forall\, i\in\mathcal{N},$$

$$0\leqslant v_{fij+1}\perp \alpha'_{ij+1}-\tau_{fij}\geqslant 0 \quad \forall\, j=1,\ldots,m-1,\ \forall\, i\in\mathcal{N},$$

$$0\leqslant \tau_{fij}\perp s_{fi}-v_{fij}+\tau_{fij}\geqslant 0 \quad \forall\, j=1,\ldots,m,\ \forall\, i\in\mathcal{N}.$$

3.1. The ISO and Transmission Fees

The ISO sets the transmission fees w to efficiently clear the market for transmission capacity. Specifically, taking w as exogenous to his problem, the ISO solves the following linear program to determine the transmission flows y to

$$\text{maximize} \quad \sum_{i\in\mathcal{N}} y_i w_i$$

$$\text{subject to} \quad \sum_{i\in\mathcal{N}} \mathrm{PDF}_{ik} y_i \leqslant T_k \quad \forall\, k\in\mathcal{K},$$

where $\mathcal{K}$ is the arc set of the electric power network and the constants T_k are the transmission capacities on the individual arcs, which we assume to be positive. The decision variables y_i represent transfers of power (in MW) by the system operator from a hub node to node i. The PDF_{ik} are the so-called power transmission distribution factors that describe how much MW flow occurs through constraint k as a result of a unit MW injection at an arbitrary hub node and a unit withdrawal at node i. In the linearized DC power flow model that is the basis for the above ISO model, these factors are constant, and the principle of superposition applies (Schweppe et al. 1988).

The above linear program (LP) has been used several times in previous models; see, e.g., Hobbs (2001) and Metzler et al. (2003). Letting λ_k be the associated dual variables of the above constraints, we can write the optimality conditions of the LP:

$$w_i = \sum_{k\in\mathcal{K}} \mathrm{PDF}_{ik}\lambda_k \quad \forall\, i\in\mathcal{N},$$

$$0\leqslant \lambda_k \perp T_k - \sum_{i\in\mathcal{N}} \mathrm{PDF}_{ik} y_i \geqslant 0 \quad \forall\, k\in\mathcal{K}.$$

These optimality conditions for the ISO of a linearized DC transmission system without resistance losses were derived by Hogan (1992) and analyzed by Boucher and Smeers (2001). A formulation with quadratic losses is presented by Chen et al. (2006). In effect, the first-order conditions show that the system operator simply sets prices to clear a market for each transmission constraint: prices can only be positive if the constraint is binding, and prices are set high enough so that the constraints are not violated.

A more general formulation would have the ISO perform an arbitrage (spot market clearing) function, as well as providing transmission services for bilateral transactions by generators. For instance, if the hub price differs from the price at another i by an amount different from w_i, then a price-taking arbitrager would find it profitable to buy (or sell) power at i and resell (or buy) it at the hub. This can be modelled by inserting unrestricted arbitrage variables in the ISO's problem with objective function coefficients equal to the price difference (Hobbs and Helman 2004), or equivalently, by defining a separate price-taking trading firm that gains revenue equal to the price difference but pays the transmission fee (Hobbs 2001, Hobbs and Pang 2004). An alternative formulation of the ISO's problem would be to reallocate power not to maximize arbitrage profits but to maximize consumer welfare (Yao et al. 2005), which can yield different decisions if marginal consumer willingness to pay for power differs from the price function because of the presence of price caps. Including an arbitrage function in the ISO's problem is straight forward. However, that would complicate the presentation below without changing the fundamental results concerning the multiplicity of solutions and difficulties with Lemke's algorithm.

3.2. The Market-Clearing Conditions

To clear the market, the transmission flows y_i must balance the net sales at each node:

$$y_i = \sum_{h\in\mathcal{F}} (s_{hi}-g_{hi}) \quad \forall\, i\in\mathcal{N}.$$

From (9), we deduce

$$y_1 = \sum_{h\in\mathcal{F}} \sum_{1\neq i\in\mathcal{N}} (g_{hi}-s_{hi}).$$

Hence,

$$\begin{aligned} T_k - \sum_{i\in\mathcal{N}} \mathrm{PDF}_{ik} y_i &= T_k - \mathrm{PDF}_{1k} y_1 - \sum_{1\neq i\in\mathcal{N}} \mathrm{PDF}_{ik} y_i \\ &= T_k - \sum_{h\in\mathcal{F}} \sum_{1\neq i\in\mathcal{N}} (\mathrm{PDF}_{ik} - \mathrm{PDF}_{1k})(s_{hi} - g_{hi}) \\ &= T_k - \sum_{h\in\mathcal{F}} \sum_{1\neq i\in\mathcal{N}} \mathrm{PDF}_{ik}(s_{hi} - g_{hi}). \end{aligned}$$

The simplification in the last line results from assuming that the hub node is node 1, in which case PDF_{1k} becomes zero. (That is, an injection and withdrawal of power both occurring at node 1 results in zero incremental flow in the network.) This convention is adopted throughout the following discussion.

3.3. The Complete LCP Formulation

Putting together the firms' optimality conditions, the ISO's problem, and the market-clearing condition, we obtain the complete formulation of the market equilibrium problem as the following LCP:

$$\begin{aligned}
&0 \leqslant s_{fi} \perp c_{f1} - P_{i0} + \beta_{i0}\left(s_{fi} + \sum_{h\in\mathcal{F}} s_{hi} - \sum_{j=1}^{m} \tau_{fij}\right) \\
&\qquad + \sum_{j=1}^{m} \beta_{ij-1}\tau_{fij} + \sum_{j=1}^{m} \beta'_{ij} v_{fij} + \sum_{k\in\mathcal{K}} \mathrm{PDF}_{ik}\lambda_k - \psi_f \\
&\qquad + \gamma_{f1} + \varphi_{fi} \geqslant 0 \quad \forall (f,i)\in\mathcal{F}\times\mathcal{N}, \\
&0 \leqslant g_{fi} \perp (c_{fi} - c_{f1}) - \sum_{k\in\mathcal{K}} \mathrm{PDF}_{ik}\lambda_k + \psi_f - \gamma_{f1} + \gamma_{fi} \geqslant 0 \\
&\qquad\qquad \forall (f,i)\in\mathcal{F}\times(\mathcal{N}\backslash\{1\}), \\
&0 \leqslant \psi_f \perp g_{f1} \equiv \sum_{i\in\mathcal{N}} s_{fi} - \sum_{1\neq i\in\mathcal{N}} g_{fi} \geqslant 0 \quad \forall f\in\mathcal{F}, \\
&0 \leqslant \lambda_k \perp T_k - \sum_{h\in\mathcal{F}} \sum_{1\neq i\in\mathcal{N}} \mathrm{PDF}_{ik}(s_{hi} - g_{hi}) \geqslant 0 \quad \forall k\in\mathcal{K}, \\
&0 \leqslant \gamma_{f1} \perp \mathrm{CAP}_{f1} - \sum_{i\in\mathcal{N}} s_{fi} + \sum_{1\neq i\in\mathcal{N}} g_{fi} \geqslant 0 \quad \forall f\in\mathcal{F}, \\
&0 \leqslant \gamma_{fi} \perp \mathrm{CAP}_{fi} - g_{fi} \geqslant 0 \quad \forall (f,i)\in\mathcal{F}\times(\mathcal{N}\backslash\{1\}), \\
&0 \leqslant \varphi_{fi} \perp \sigma_i - \sum_{h\in\mathcal{F}} s_{hi} \geqslant 0 \quad \forall (f,i)\in\mathcal{F}\times\mathcal{N}, \\
&0 \leqslant v_{fi1} \perp \alpha'_{i1} - \left(S_i - \sum_{j=1}^{m} \tau_{fij}\right) \geqslant 0 \quad \forall (f,i)\in\mathcal{F}\times\mathcal{N}, \\
&0 \leqslant v_{fij+1} \perp \alpha'_{ij+1} - \tau_{fij} \geqslant 0 \\
&\qquad\qquad \forall j = 1,\ldots,m-1,\ \forall (f,i)\in\mathcal{F}\times\mathcal{N}, \\
&0 \leqslant \tau_{fij} \perp s_{fi} - v_{fij} + \tau_{fij} \geqslant 0 \\
&\qquad\qquad \forall j = 1,\ldots,m,\ \forall (f,i)\in\mathcal{F}\times\mathcal{N}.
\end{aligned} \tag{10}$$

The proof of Proposition 1 shows that as long as the last three conditions hold, we must have

$$\tau_{fi0} \equiv S_i - \sum_{j=1}^{m} \tau_{fij} \geqslant 0 \quad \forall (f,i)\in\mathcal{F}\times\mathcal{N}, \tag{11}$$

which recovers Equation (7) that is used to express the regional sales variable S_i.

4. Solution by Lemke's Method

The system (10) can easily be written compactly as the LCP:

$$0 \leqslant z \perp y \equiv q + Mz \geqslant 0 \tag{12}$$

for some vector q matrix M whose entries are not difficult to identify (details are omitted). The variable z concatenates several groups of variables: (s, g) the firms' primal variables, (ψ) the special multipliers associated with the constraints $\sum_{i\in\mathcal{N}} s_{fi} - \sum_{1\neq i\in\mathcal{N}} g_{fi} \geqslant 0$, (m) the multipliers of the ISO's transmission constraints and those of the firms' decoupled constraints, (φ) the multipliers of the firms' common constraints, and (a) the auxiliary variables of the nodal price functions:

(s) $\{s_{fi}\colon (f,i)\in\mathcal{F}\times\mathcal{N}\}$,
(g) $\{g_{fi}\colon (f,i)\in\mathcal{F}\times(\mathcal{N}\backslash\{1\})\}$,
(ψ) $\{\psi_f\colon f\in\mathcal{F}\}$,
(m) $\{\lambda_k\colon k\in\mathcal{K}\}$ and $\{\gamma_{fi}\colon (f,i)\in\mathcal{F}\times\mathcal{N}\}$,
(φ) $\{\varphi_{fi}\colon (f,i)\in\mathcal{F}\times\mathcal{N}\}$,
(a) $\{v_{fij},\ \tau_{fij}\colon (f,i)\in\mathcal{F}\times\mathcal{N},\ j=1,\ldots,m\}$.

Partitioned accordingly, the constant vector q and the matrix M are of the form

$$q \equiv \begin{pmatrix} q^s \\ q^g \\ 0 \\ q^m \\ q^\varphi \\ q^a \end{pmatrix},$$

$$M \equiv \left[\begin{array}{cc|cccc} M^{ss} & 0 & M^{s\psi} & M^{sm} & M^{s\varphi} & M^{sa} \\ 0 & 0 & M^{g\psi} & M^{gm} & 0 & 0 \\ \hline -(M^{s\psi})^T & -(M^{g\psi})^T & 0 & 0 & 0 & 0 \\ -(M^{sm})^T & -(M^{gm})^T & 0 & 0 & 0 & 0 \\ -N^{\varphi s} & 0 & 0 & 0 & 0 & 0 \\ M^{as} & 0 & 0 & 0 & 0 & M^{aa} \end{array}\right]. \tag{13}$$

It is not difficult to see that the principal submatrix M^{ss} is symmetric positive definite. Hence, it follows that the principal submatrix

$$\widetilde{M} \equiv \begin{bmatrix} M^{ss} & 0 & M^{s\psi} & M^{sm} \\ 0 & 0 & M^{g\psi} & M^{gm} \\ -(M^{s\psi})^T & -(M^{g\psi})^T & 0 & 0 \\ -(M^{sm})^T & -(M^{gm})^T & 0 & 0 \end{bmatrix}$$

is positive semidefinite (albeit not symmetric). The submatrix $M^{s\varphi}$ is clearly nonnegative. Because for each fixed i, the scalars β_{ij} satisfy $\beta_{im} > \cdots > \beta_{i0}$, the matrix M^{sa} is nonnegative. Based on these preliminary facts, Proposition 2 establishes that the matrix M in the LCP (12) is *semicopositive* (Facchinei and Pang 2003) (originally called *semimonotone* in Cottle et al. 1992); i.e.,

$$0 \neq z \geqslant 0 \Rightarrow \exists i \quad \text{such that } z_i \neq 0 \text{ and } z_i(Mz)_i \geqslant 0.$$

PROPOSITION 2. *The matrix M in* (12) *of the Nash equilibrium problem is semicopositive.*

PROOF. The matrix M has the form

$$M \equiv \begin{bmatrix} \widetilde{M} & C \\ E & \widehat{M}^{aa} \end{bmatrix}, \quad \text{where } \widehat{M}^{aa} \equiv \begin{bmatrix} 0 & 0 \\ 0 & M^{aa} \end{bmatrix}$$

for some suitable matrices C and E, with C being nonnegative. We claim that M^{aa} is semicopositive; to demonstrate this, let v_{fij} and τ_{fij} be given nonnegative scalars, not all equal to zero. If $v_{fi1} > 0$ for some pair (f, i), then clearly $v_{fi1} \sum_{j=1}^{m} \tau_{fij} \geqslant 0$. Hence, without loss of generality, assume that $v_{fi1} = 0$ for all pairs (f, i). If $\tau_{fi1} > 0$ for some (f, i), then $\tau_{fi1}(-v_{fi1} + \tau_{fi1}) > 0$. Therefore, we may assume, without loss of generality, that $\tau_{fi1} = 0$ for all pairs (f, i). If $v_{fi2} > 0$ for some (f, i), then $v_{fi2}(-\tau_{fi1}) = 0$. Hence, without loss of generality, we may assume that $v_{fi2} = 0$ for all pairs (f, i), which then implies $\tau_{fi2}(-v_{fi1} + \tau_{fi2}) = (\tau_{fi2})^2$. Continuing the argument inductively, we can eventually show that M^{aa} is semicopositive.

To complete the proof, let z be a nonzero, nonnegative vector whose components are partitioned in six groups $(s, g, \psi, m, \varphi, a)$ as described above. If any one of the first four groups (s, g, ψ, m) of variables has a nonzero component, then the semicopositivity of M follows easily from the positive semidefiniteness of $\widetilde{M}$ and the nonnegativity of C. If the variables in the four groups (s, g, ψ, m) are all zero, and if φ_{fi} is positive for some pair (f, i), then the semicopositivity of M follows readily. Finally, if the variables in the five groups (s, g, ψ, m, φ) are all zero, then the semicopositivity of M follows from that of the principal submatrix M^{aa}. □

In spite of the favorable property of M, the example below shows that Lemke's method is not guaranteed to successfully compute a complementarity solution of the Nash equilibrium model formulated as the LCP (10).

EXAMPLE 2. Consider a three-firm, two-node problem with the nodal price functions being linear and given by

$$p_1(S_1) = 1 - \tfrac{1}{2} S_1 \quad \text{and} \quad p_2(S_2) = 2 - \tfrac{1}{4} S_2.$$

There are two common coupling constraints:

$$S_1 \leqslant 3/4 \quad \text{and} \quad S_2 \leqslant 2.$$

The unit production costs are as follows:

$$(c_{11}, c_{12}) = (0.5, 1), \quad (c_{21}, c_{22}) = (0.5, 1.5), \quad \text{and}$$
$$(c_{31}, c_{32}) = (1.5, 0.5).$$

The generation capacities are all equal to one unit; i.e., $\text{CAP}_{fi} = 1$ for all $f = 1, 2, 3$ and $i = 1, 2$. The matrix M defining the LCP (10) is of order 21×21. When this LCP is solved by Lemke's method (1965), it turns out that there is a significant number of degenerate pivots. We used a MATLAB code written by Michael Ferris that resolves such degeneracy by means of a tie breaking rule based on the random number generator RAND within MATLAB. For most of the runs, degeneracy was not an issue and the method successfully terminated with a complementary solution to the problem. Nevertheless, there were runs where termination occurred on a secondary ray. One such failed run involves the following sequence of leaving variables: y_2, y_4, y_9, y_{21}, y_{16}, y_3, y_{11}, y_1, z_{16}, y_{18}, z_3, y_{16}, z_1, z_{11}, y_7, y_{15}, and ray termination.

Even worse performance resulted when we changed the unit production costs to $c_{fi} = 3$ for all $f = 1, 2, 3$ and $i = 1, 2$. Again, many pivots are degenerate, and none of the runs is successful, although the problem has a complementary solution.

To understand the cause of the failure of the pivoting algorithm, we consider a simpler problem with two players whose optimization problems are as follows:

$$\begin{aligned} &\underset{x_1}{\text{maximize}} && x_1\left(1 - \tfrac{1}{2}x_1 - \tfrac{1}{2}x_2\right) \\ &\text{subject to} && x_1 + x_2 \leqslant 1 \quad \text{and} \\ &&& x_1 \geqslant 0, \\ &\underset{x_2}{\text{maximize}} && x_2\left(2 - \tfrac{1}{2}x_1 - \tfrac{1}{2}x_2\right) \\ &\text{subject to} && x_1 + x_2 \leqslant 1 \quad \text{and} \\ &&& x_2 \geqslant 0. \end{aligned}$$

Note the common constraint $x_1 + x_2 \leqslant 1$. The LCP formulation of this equilibrium problem is as follows:

$$\begin{aligned} &0 \leqslant -1 + x_1 + \tfrac{1}{2}x_2 + \varphi_1 \perp x_1 \geqslant 0 \\ &0 \leqslant -2 + \tfrac{1}{2}x_1 + x_2 + \varphi_2 \perp x_2 \geqslant 0 \\ &0 \leqslant 1 - x_1 - x_2 \perp \varphi_1 \geqslant 0 \\ &0 \leqslant 1 - x_1 - x_2 \perp \varphi_2 \geqslant 0. \end{aligned}$$

To initiate Lemke's algorithm (1965), we augment the LCP by adding an artificial variable z_0 and an artificial column of all ones:

$$\begin{aligned} &0 \leqslant y_1 = -1 + z_0 + x_1 + \tfrac{1}{2}x_2 + \varphi_1 \perp x_1 \geqslant 0, \\ &0 \leqslant y_2 = -2 + z_0 + \tfrac{1}{2}x_1 + x_2 + \varphi_2 \perp x_2 \geqslant 0, \\ &0 \leqslant \mu_1 = 1 + z_0 - x_1 - x_2 \perp \varphi_1 \geqslant 0, \\ &0 \leqslant \mu_2 = 1 + z_0 - x_1 - x_2 \perp \varphi_2 \geqslant 0. \end{aligned}$$

In the first pivot, z_0 enters the basis driving out y_2. The next entering variable is thus x_2. At this point, there is a tie in the ratio test, with both μ_1 and μ_2 being the candidate variables to become nonbasic. The system is as follows:

$$0 \leqslant y_1 = 1 + y_2 + \tfrac{1}{2}x_1 - \tfrac{1}{2}x_2^{\uparrow} + \varphi_1 - \varphi_2 \perp x_1 \geqslant 0,$$

$$z_0 = 2 + y_2 - \tfrac{1}{2}x_1 - x_2^{\uparrow} - \varphi_2,$$

$$0 \leqslant \mu_1 = 3 + y_2 - 1.5x_1 - 2x_2^{\uparrow} - \varphi_2 \perp \varphi_1 \geqslant 0,$$

$$0 \leqslant \mu_2 = 3 + y_2 - 1.5x_1 - 2x_2^{\uparrow} - \varphi_2 \perp \varphi_2 \geqslant 0.$$

If μ_1 is chosen to be the leaving variable, then φ_1 becomes the next entering variable, and ray termination occurs. Instead, if μ_2 is chosen to be the leaving variable, then the method successfully computes the equilibrium solution of $(x_1, x_2) = (1, 0)$ and $(\varphi_1, \varphi_2) = (0, 3/2)$. Note that the two multipliers φ_1 and φ_2 are not equal in this solution. The problem has an equilibrium solution $(x_1, x_2) = (0, 1)$ with equal multipliers $\varphi_1 = \varphi_2 = 1$. Finally, every solution to the problem must have $\varphi_2 > 0$. □

4.1. A Restricted-Multiplier Formulation

The culprit in the failure of Lemke's method (1965) for solving the LCP (10) is the fact that while the joint sales cap constraint (2) is the same for all firms $f \in \mathcal{F}$, we have associated with it a multiplier φ_{fi} that is firm-dependent. Whereas this is theoretically correct, the generality leads to an LCP that is difficult to solve. Therefore, as a remedy, we propose a restricted LCP whereby the multiplier associated with the above constraint depends only on the region i and applies to all firms. Specifically, the restricted LCP is obtained from (10) by setting $\varphi_{fi} = \varphi_i$ for all $f \in \mathcal{F}$:

$$0 \leqslant s_{fi} \perp c_{f1} - P_{i0} + \beta_{i0}\left(s_{fi} + \sum_{h\in\mathcal{F}} s_{hi} - \sum_{j=1}^{m} \tau_{fij}\right) + \sum_{j=1}^{m} \beta_{ij-1}\tau_{fij} + \sum_{j=1}^{m} \beta'_{ij} v_{fij} + \sum_{k\in\mathcal{K}} \mathrm{PDF}_{ik}\lambda_k - \psi_f + \gamma_{f1} + \varphi_i \geqslant 0 \quad \forall (f,i) \in \mathcal{F} \times \mathcal{N},$$

$$0 \leqslant g_{fi} \perp (c_{fi} - c_{f1}) - \sum_{k\in\mathcal{K}} \mathrm{PDF}_{ik}\lambda_k + \psi_f - \gamma_{f1} + \gamma_{fi} \geqslant 0 \quad \forall (f,i) \in \mathcal{F} \times (\mathcal{N}\backslash\{1\}),$$

$$0 \leqslant \psi_f \perp g_{f1} \equiv \sum_{i\in\mathcal{N}} s_{fi} - \sum_{1\neq i\in\mathcal{N}} g_{fi} \geqslant 0 \quad \forall f \in \mathcal{F},$$

$$0 \leqslant \lambda_k \perp T_k - \sum_{h\in\mathcal{F}} \sum_{1\neq i\in\mathcal{N}} \mathrm{PDF}_{ik}(s_{hi} - g_{hi}) \geqslant 0 \quad \forall f \in \mathcal{K},$$

$$0 \leqslant \gamma_{f1} \perp \mathrm{CAP}_{f1} - \sum_{i\in\mathcal{N}} s_{fi} + \sum_{1\neq i\in\mathcal{N}} g_{fi} \geqslant 0 \quad \forall f \in \mathcal{F},$$

$$0 \leqslant \gamma_{fi} \perp \mathrm{CAP}_{fi} - g_{fi} \geqslant 0 \quad \forall (f,i) \in \mathcal{F} \times (\mathcal{N}\backslash\{1\}),$$

$$0 \leqslant \varphi_i \perp \sigma_i - \sum_{h\in\mathcal{F}} s_{hi} \geqslant 0 \quad \forall i \in \mathcal{N},$$

$$0 \leqslant v_{fi1} \perp \alpha'_{i1} - \left(S_i - \sum_{j=1}^{m} \tau_{fij}\right) \geqslant 0 \quad \forall (f,i) \in \mathcal{F} \times \mathcal{N},$$

$$0 \leqslant v_{fij+1} \perp \alpha'_{ij+1} - \tau_{fij} \geqslant 0 \quad \forall j = 1, \ldots, m-1,\ \forall (f,i) \in \mathcal{F} \times \mathcal{N},$$

$$0 \leqslant \tau_{fij} \perp s_{fi} - v_{fij} + \tau_{fij} \geqslant 0 \quad \forall j = 1, \ldots, m,\ \forall (f,i) \in \mathcal{F} \times \mathcal{N}. \quad (14)$$

Originally discussed in Harker (1991) and subsequently applied to transmission constraints by Wei and Smeers (1999), the above restricted-multiplier formulation is applicable to all generalized Nash games where the players have the same joint constraint (also called "coupled constraints" by some authors). For a recent application of such a game in electric power markets, see Contreras et al. (2004). In these cited papers on joint constraints (as well as in others, such as Berridge and Krawczyk 1997 and Krawczyk and Uryasev 2000), there is no discussion on either piecewise objectives or the applicability of Lemke's method. Therefore, our contribution distinguishes itself from the cited works in two major ways: From a modeling point of view, our model includes both features (piecewise objectives and joint constraints); algorithmically, we establish the successful termination of Lemke's method.

In the case of the sales-cap constraints (2), the economic interpretation of the restricted formulation is that a limited resource represented by σ_i is rationed among the producers by an auction or market process that produces a single clearing price. This clearing price is, in effect, treated as exogenous by all producers. Other joint constraints can be handled by the same approach as long as they are common to all firms' problems. See the concluding remarks (§6) for further comments on the restricted formulation.

In the special case where the joint constraint set is the same for all producers and coincides with the transmission constraint set in the ISO's problem (§3.1), the problem can be simplified as follows: Delete the joint constraint set and associated dual variables from each producer's problem so that those constraints appear only in the ISO's problem. The resulting model was proposed in Hobbs (2001) and analyzed in Metzler et al. (2003). However, the resulting solution is, in general, only a subset of the many possible solutions of the original problem. For some network configurations (such as the two node example in Figure 2 and §5, and the three node case considered by Oren 1997), an important possible solution that would be excluded is one in which the w are zero and the multipliers of the joint constraints are positive and possibly different for the various producers. The zero w is important from a practical standpoint because it means that the ISO receives no revenue from providing transmission services, even if the transmission constraints are binding. This, as Oren (1997) explains, is a type of market power in which electricity producers exploit the network operator. Oren's model differs from the one of this paper; it is instead an example of the MPEC/EPEC formulation referred to in §1 in which generators are assumed to sell power only at the nodes at which they generate, with the system operator purchasing the power there and then reselling it to consumers elsewhere. Nevertheless, the prices, profits, and generation amounts obtained in those of his solutions that have zero transmission prices and a multiplicity of possible allocations of transmission capacity among producers are also equilibria in our model.

To initiate Lemke's method on the LCP (14), we consider the following augmented system with z_0 as the artificial variable and positive auxiliary vectors d^s, d^g, and d^ψ added to the first three conditions in (14):

$$0 \leqslant s_{fi} \perp c_{f1} - P_{i0} + z_0 d^s_{fi} + \beta_{i0}\left(s_{fi} + \sum_{h\in\mathcal{F}} s_{hi} - \sum_{j=1}^{m}\tau_{fij}\right)$$
$$+ \sum_{j=1}^{m}\beta_{ij-1}\tau_{fij} + \sum_{j=1}^{m}\beta'_{ij}v_{fij} + \sum_{k\in\mathcal{K}}\mathrm{PDF}_{ik}\lambda_k - \psi_f$$
$$+ \gamma_{f1} + \varphi_i \geqslant 0 \quad \forall (f,i)\in\mathcal{F}\times\mathcal{N},$$
$$0 \leqslant g_{fi} \perp c_{fi} - c_{f1} + z_0 d^g_{fi} - \sum_{k\in\mathcal{K}}\mathrm{PDF}_{ik}\lambda_k + \psi_f - \gamma_{f1} + \gamma_{fi} \geqslant 0 \quad \forall (f,i)\in\mathcal{F}\times(\mathcal{N}\backslash\{1\}),$$
$$0 \leqslant \psi_f \perp z_0 d^\psi_f + \sum_{i\in\mathcal{N}} s_{fi} - \sum_{1\neq i\in\mathcal{N}} g_{fi} \geqslant 0 \quad \forall f\in\mathcal{F},$$
$$0 \leqslant \lambda_k \perp T_k - \sum_{h\in\mathcal{F}}\sum_{1\neq i\in\mathcal{N}}\mathrm{PDF}_{ik}(s_{hi} - g_{hi}) \geqslant 0 \quad \forall k\in\mathcal{K},$$
$$0 \leqslant \gamma_{f1} \perp \mathrm{CAP}_{f1} - \sum_{i\in\mathcal{N}} s_{fi} + \sum_{1\neq i\in\mathcal{N}} g_{fi} \geqslant 0 \quad \forall f\in\mathcal{F},$$
$$0 \leqslant \gamma_{fi} \perp \mathrm{CAP}_{fi} - g_{fi} \geqslant 0 \quad \forall (f,i)\in\mathcal{F}\times(\mathcal{N}\backslash\{1\}),$$
$$0 \leqslant \varphi_i \perp \sigma_i - \sum_{h\in\mathcal{F}} s_{hi} \geqslant 0 \quad \forall i\in\mathcal{N},$$
$$0 \leqslant v_{fi1} \perp \alpha'_{i1} - \left(\sum_{h\in\mathcal{F}} s_{hi} - \sum_{j=1}^{m}\tau_{fij}\right) \geqslant 0 \quad \forall (f,i)\in\mathcal{F}\times\mathcal{N},$$
$$0 \leqslant v_{fij+1} \perp \alpha'_{ij+1} - \tau_{fij} \geqslant 0 \quad \forall j=1,\ldots,m-1,\ \forall (f,i)\in\mathcal{F}\times\mathcal{N},$$
$$0 \leqslant \tau_{fij} \perp s_{fi} - v_{fij} + \tau_{fij} \geqslant 0 \quad \forall j=1,\ldots,m,\ \forall (f,i)\in\mathcal{F}\times\mathcal{N}.$$

4.2. Proof of Successful Termination

Referring to the structure (13) of the pair (q, M) in (10), we first establish the following preliminary result, whose proof is similar to that of Proposition 2.

LEMMA 2. *Let*

$$q^0 \equiv \begin{pmatrix} d^s \\ d^g \\ d^\psi \\ d^m \\ \sigma \\ q^a \end{pmatrix} \quad \text{with } d^m \equiv \begin{pmatrix} T \\ \mathrm{CAP} \end{pmatrix},$$

where d^s, d^g, and d^ψ are arbitrary positive vectors. The only solution z to the system

$$0 \leqslant z \perp q^0 + Mz \geqslant 0$$

is the zero vector.

PROOF. By the remark made at the end of § 3.3, it follows that (11) holds. Consequently, we deduce that

$$0 = \sum_{(f,i)\in\mathcal{F}\times\mathcal{N}} s_{fi}\left[d^s_{fi} + \beta_{i0}(s_{fi} + \tau_{fi0}) + \sum_{j=1}^{m}\beta_{ij-1}\tau_{fij} + \sum_{j=1}^{m}\beta'_{ij}v_{fij} - \psi_f + \gamma_{f1} + \varphi_i + \sum_{k\in\mathcal{K}}\mathrm{PDF}_{ik}\lambda_k\right]$$
$$+ \sum_{(f,i)\in\mathcal{F}\times(\mathcal{N}\backslash\{1\})} g_{fi}\left[d^g_{fi} - \sum_{k\in\mathcal{K}}\mathrm{PDF}_{ik}\lambda_k + \psi_f - \gamma_{f1} + \gamma_{fi}\right] + \sum_{f\in\mathcal{F}}\psi_f\left[d^\psi_f + \sum_{i\in\mathcal{N}} s_{fi} - \sum_{1\neq i\in\mathcal{N}} g_{fi}\right]$$
$$+ \sum_{k\in\mathcal{K}}\lambda_k\left[T_k - \sum_{h\in\mathcal{F}}\sum_{1\neq i\in\mathcal{N}}\mathrm{PDF}_{ik}(s_{hi} - g_{hi})\right]$$
$$+ \sum_{f\in\mathcal{F}}\gamma_{f1}\left[\mathrm{CAP}_{f1} - \sum_{i\in\mathcal{N}} s_{fi} + \sum_{1\neq i\in\mathcal{N}} g_{fi}\right]$$
$$+ \sum_{(f,i)\in\mathcal{F}\times(\mathcal{N}\backslash\{1\})}\gamma_{fi}(\mathrm{CAP}_{fi} - g_{fi})$$
$$\geqslant \sum_{(f,i)\in\mathcal{F}\times\mathcal{N}} s_{fi}(d^s_{fi} + \beta_{i0}s_{fi}) + \sum_{(f,i)\in\mathcal{F}\times(\mathcal{N}\backslash\{1\})} g_{fi}d^g_{fi} + \sum_{f\in\mathcal{F}}\psi_f d^\psi_f + \sum_{k\in\mathcal{K}}\lambda_k T_k + \sum_{(f,i)\in\mathcal{F}\times\mathcal{N}}\gamma_{fi}\mathrm{CAP}_{fi}.$$

Because the constants d^s_{fi}, d^g_{fi}, d^ψ_f, T_k, and CAP_{fi} are all positive, it follows that

$$s_{fi} = g_{fi} = \psi_f = \lambda_k = \gamma_{fi} = 0 \quad \forall k\in\mathcal{K},\ (f,i)\in\mathcal{F}\times\mathcal{N}.$$

Because

$$0 = v_{fi1}\left(\alpha'_{i1} - S_i + \sum_{j=1}^{m}\tau_{fij}\right) = v_{fi1}\left(\alpha'_{i1} + \sum_{j=1}^{m}\tau_{fij}\right),$$

it follows that $v_{fi1} = 0$, and thus $\tau_{fi1} = 0$ for all $(f,i)\in\mathcal{F}\times\mathcal{N}$. Inductively, we can deduce that $v_{fij+1} = \tau_{fij+1} = 0$ for all $(f,i)\in\mathcal{F}\times\mathcal{N}$ and $j = 1,\ldots,m$. Because

$$0 \leqslant \varphi_i \perp \sigma_i - S_i = \sigma_i > 0,$$

it follows that $\varphi_i = 0$ for all $i\in\mathcal{N}$. □

The above proposition does not imply that the matrix defining the overall LCP (14) is an "R_0 matrix" (Cottle et al. 1992) (although it must be semicopositive). Therefore, the successful termination of Lemke's method for solving this LCP does not follow from known results. A detailed proof of success proceeds in the usual way, i.e., by assuming that the method terminates on a secondary ray. It then follows that there exist (z_0^*, z^*) with $z^* \neq 0$ and $(\tilde{z}_0, \tilde{z}) \neq 0$ (these are obvious shorthand representations for the actual variables as in the compact formulation (12)) such that, for all $\rho \geqslant 0$,

$$\begin{aligned}
0 \leqslant s_{fi}^* + \rho\tilde{s}_{fi} \perp c_{f1} - P_{i0} + (z_0^* + \rho\tilde{z}_0)d_{fi}^s \\
+ \beta_{i0}\Bigg(s_{fi}^* + \rho\tilde{s}_{fi} + \sum_{h\in\mathcal{F}}(s_{hi}^* + \rho\tilde{s}_{hi}) \\
- \sum_{j=1}^m(\tau_{fij}^* + \rho\tilde{\tau}_{fij})\Bigg) \\
+ \sum_{j=1}^m \beta_{ij-1}(\tau_{fij}^* + \rho\tilde{\tau}_{fij}) + \sum_{j=1}^m \beta'_{ij}(\upsilon_{fij}^* + \rho\tilde{\upsilon}_{fij}) \\
+ \sum_{k\in\mathcal{K}} \mathrm{PDF}_{ik}(\lambda_k^* + \rho\tilde{\lambda}_k) - (\psi_f^* + \rho\tilde{\psi}_f) \\
+ (\gamma_{f1}^* + \rho\tilde{\gamma}_{f1}) + (\varphi_i^* + \rho\tilde{\varphi}_i) \geqslant 0,
\end{aligned}$$

$$\begin{aligned}
0 \leqslant g_{fi}^* + \rho\tilde{g}_{fi} \perp c_{fi} - c_{f1} + (z_0^* + \rho\tilde{z}_0)d_{fi}^g \\
- \sum_{k\in\mathcal{K}} \mathrm{PDF}_{ik}(\lambda_k^* + \rho\tilde{\lambda}_k) + (\psi_f^* + \rho\tilde{\psi}_f) \\
- (\gamma_{f1}^{*,+} + \rho\tilde{\gamma}_{f1}) + (\gamma_{fi}^* + \rho\tilde{\gamma}_{fi}) \geqslant 0,
\end{aligned}$$

$$0 \leqslant \psi_f^* + \rho\tilde{\psi}_f \perp (z_0^* + \rho\tilde{z}_0)d_f^\psi + \sum_{i\in\mathcal{N}}(s_{fi}^* + \rho\tilde{s}_{fi}) - \sum_{1\neq i\in\mathcal{N}}(g_{fi}^* + \rho\tilde{g}_{fi}) \geqslant 0,$$

$$0 \leqslant \lambda_k^* + \rho\tilde{\lambda}_k \perp T_k - \sum_{h\in\mathcal{F}}\sum_{1\neq i\in\mathcal{N}} \mathrm{PDF}_{ik}(s_{hi}^* + \rho\tilde{s}_{hi} - g_{hi}^* - \rho\tilde{g}_{hi}) \geqslant 0,$$

$$0 \leqslant \gamma_{f1}^* + \rho\tilde{\gamma}_{f1} \perp \mathrm{CAP}_{f1} - \sum_{i\in\mathcal{N}}(s_{fi}^* + \rho\tilde{s}_{fi}) + \sum_{1\neq i\in\mathcal{N}}(g_{fi}^* + \rho\tilde{g}_{fi}) \geqslant 0,$$

$$0 \leqslant \gamma_{fi}^* + \rho\tilde{\gamma}_{fi} \perp \mathrm{CAP}_{fi} - (g_{fi}^* + \rho\tilde{g}_{fi}) \geqslant 0,$$

$$0 \leqslant \varphi_i^* + \rho\tilde{\varphi}_i \perp \sigma_i - \sum_{h\in\mathcal{F}}(s_{hi}^* + \rho\tilde{s}_{hi}) \geqslant 0,$$

$$0 \leqslant \upsilon_{fi1}^* + \rho\tilde{\upsilon}_{fi1} \perp \alpha'_{i1} - \Bigg[\sum_{h\in\mathcal{F}}(s_{hi}^* + \rho\tilde{s}_{hi}) - \sum_{j=1}^m(\tau_{fij}^* + \rho\tilde{\tau}_{fij})\Bigg] \geqslant 0,$$

$$0 \leqslant \upsilon_{fij+1}^* + \rho\tilde{\upsilon}_{fij+1} \perp \alpha'_{ij+1} - (\tau_{fij}^* + \rho\tilde{\tau}_{fij}) \geqslant 0,$$

$$0 \leqslant \tau_{fij}^* + \rho\tilde{\tau}_{fij} \perp s_{fi}^* + \rho\tilde{s}_{fi} - (\upsilon_{fij}^* + \rho\tilde{\upsilon}_{fij}) + (\tau_{fij}^* + \rho\tilde{\tau}_{fij}) \geqslant 0.$$

The following proof aims at deriving a contradiction by showing that the pair $(\tilde{z}_0, \tilde{z})$ must be zero. Because the above system holds for all $\rho > 0$, it follows from Lemma 2 that $\tilde{z}_0 = 0$. Because

$$\mathrm{CAP}_{fi} - (g_{fi}^* + \rho\tilde{g}_{fi}) \geqslant 0 \quad \forall \rho \geqslant 0,$$

it follows that $\tilde{g}_{fi} = 0$ for all $f \in \mathcal{F}$ and all $i \in \mathcal{N}\backslash\{1\}$. Because

$$\mathrm{CAP}_{f1} - \sum_{i\in\mathcal{N}}(s_{fi}^* + \rho\tilde{s}_{fi}) + \sum_{1\neq i\in\mathcal{N}} g_{fi}^* \geqslant 0 \quad \forall \rho \geqslant 0,$$

it follows that $\tilde{s}_{fi} = 0$ for all $f \in \mathcal{F}$ and all $i \in \mathcal{N}$. Because

$$\alpha'_{ij+1} - (\tau_{fij}^* + \rho\tilde{\tau}_{fij}) \geqslant 0 \quad \forall \rho \geqslant 0,$$

it follows that $\tilde{\tau}_{fij} = 0$ for all $f \in \mathcal{F}$, $i \in \mathcal{N}$, and all $j = 1, \ldots, m-1$. Adding the two inequalities

$$\alpha'_{ij+1} - \tau_{fij}^* \geqslant 0 \quad \text{and}$$

$$s_{fi}^* - (\upsilon_{fij}^* + \rho\tilde{\upsilon}_{fij}) + \tau_{fij}^* \geqslant 0 \quad \forall \rho \geqslant 0,$$

which holds for all $f \in \mathcal{F}$, $i \in \mathcal{N}$, and all $j = 1, \ldots, m-1$, we deduce $\tilde{\upsilon}_{fij} = 0$ for all such triples (f, i, j). Because

$$0 \leqslant \tilde{\tau}_{fim} \perp -\tilde{\upsilon}_{fim} + \tilde{\tau}_{fim} \geqslant 0,$$

it follows that $\tilde{\tau}_{fim} = \tilde{\upsilon}_{fim}$. Similar to the last part in the proof of Proposition 1, we can deduce

$$\begin{aligned}
0 &\leqslant \sum_{h\in\mathcal{F}}(s_{hi}^* + \rho\tilde{s}_{hi}) - \sum_{j=1}^m(\tau_{fij}^* + \rho\tilde{\tau}_{fij}) \\
&= \sum_{h\in\mathcal{F}} s_{hi}^* - \sum_{j=1}^m \tau_{fij}^* - \rho\tilde{\tau}_{fim},
\end{aligned}$$

which easily implies $\tilde{\tau}_{fim} = 0$ for all $f \in \mathcal{F}$ and $i \in \mathcal{N}$.

Taking into account what has been shown so far, we obtain

$$\begin{aligned}
0 = &\sum_{(f,i)\in\mathcal{F}\times\mathcal{N}} s_{fi}^*[c_{f1} - P_{i0} + z_0^* d_{fi}^s] \\
&+ \sum_{(f,i)\in\mathcal{F}\times\mathcal{N}} s_{fi}^*\Bigg[\beta_{i0}\Bigg(s_{fi}^* + \sum_{h\in\mathcal{F}} s_{hi}^* - \sum_{j=1}^m \tau_{fij}^*\Bigg) \\
&\qquad + \sum_{j=1}^m \beta_{ij-1}\tau_{fij}^* + \sum_{j=1}^m \beta'_{ij}\upsilon_{fij}^* + \varphi_i^* + \rho\tilde{\varphi}_i\Bigg] \\
&+ \sum_{(f,i)\in\mathcal{F}\times(\mathcal{N}\backslash\{1\})} g_{fi}^*[c_{fi} - c_{f1} + z_0^* d_{fi}^g] \\
&+ \sum_{f\in\mathcal{F}}(\psi_f^* + \rho\tilde{\psi}_f)z_0^* d_{f1}^g + \sum_{k\in\mathcal{A}}(\lambda_k^* + \rho\tilde{\lambda}_k)T_k \\
&+ \sum_{(f,i)\in\mathcal{F}\times\mathcal{N}}(\gamma_{fi}^* + \rho\tilde{\gamma}_{fi})\mathrm{CAP}_{fi}.
\end{aligned}$$

Again, because this has to hold for all $\rho > 0$, it follows that $\tilde{\lambda}_k = s_{fi}^*\tilde{\varphi}_i = \tilde{\psi}_{f1} = \tilde{\gamma}_{fi} = 0$ for all $(f, i) \in \mathcal{F} \times \mathcal{N}$. Therefore, all the tilde variables have been shown to equal zero, except for the $\tilde{\varphi}_i$ variables. But we must have the latter variables equal to zero, too, because

$$0 = \tilde{\varphi}_i\Bigg(\sigma_i - \sum_{h\in\mathcal{F}} s_{hi}^*\Bigg) = \tilde{\varphi}_i\sigma_i$$

and $\sigma_i > 0$. Therefore, $\tilde{z}$ and $\tilde{z}_0$ are both zero, which is a contradiction. Summarizing the above proof, we have therefore established Theorem 1.

THEOREM 1. *Under a standard nondegeneracy assumption, Lemke's method will successfully compute a solution to the restricted LCP* (14).

5. A Numerical Example

This example includes both price caps and a joint (transmission) constraint, and illustrates several economic implications of these complications. The network topology is shown in Figure 2. The nodal prices are equal to $\min(0.25, 1-S_i)$ for $i = 1, 2$. Thus, there is a price cap of 0.25 imposed as a market power mitigation measure: Price at neither node is allowed to exceed 0.25. In terms of the representation (3), we have $m = 1$, and for $i = 1, 2$,

$$P_{i0} = 0.25, \quad \alpha_{i1} = 0.75, \quad \beta_{i0} = 0, \quad \text{and} \quad \beta_{i1} = 1,$$

which yield $\beta'_{i1} = 1$ and $\alpha'_{i1} = 0.75$. The transmission capacity T between the nodes is 0.5. (The subscript k on that constraint is omitted, because there is only one such constraint.) The resulting transmission constraint, $s_{A2} + s_{B2} \leqslant 0.5$, is a joint constraint that is included not only in the ISO's problem but also in the profit maximization problems for firms A and B. Thus, for instance, given that the game is Cournot, firm A can choose any sales $0 \leqslant s_{A2} \leqslant 0.5 - s_{B2}$. Firms A and B each have one power plant (at node 1) without a capacity upper bound, but they have different marginal costs: $c_{A1} = 0.1$ and $c_{B1} = 0.0$. The LCP (10) for this example is as follows:

$$0 \leqslant s_{A1} \perp -0.15 + v_{A11} - \psi_A \geqslant 0,$$
$$0 \leqslant s_{A2} \perp -0.15 + v_{A21} + \lambda - \psi_A + \varphi_{A2} \geqslant 0,$$
$$0 \leqslant s_{B1} \perp -0.25 + v_{B11} - \psi_B \geqslant 0,$$
$$0 \leqslant s_{A2} \perp -0.25 + v_{B21} + \lambda - \psi_B + \varphi_{B2} \geqslant 0,$$
$$0 \leqslant \psi_A \perp g_A \equiv s_{A1} + s_{A2} \geqslant 0,$$
$$0 \leqslant \psi_B \perp g_B \equiv s_{B1} + s_{B2} \geqslant 0,$$
$$0 \leqslant \lambda \perp 0.5 - s_{A2} - s_{B2} \geqslant 0,$$
$$0 \leqslant \varphi_{A2} \perp 0.5 - s_{A2} - s_{B2} \geqslant 0,$$
$$0 \leqslant \varphi_{B2} \perp 0.5 - s_{A2} - s_{B2} \geqslant 0,$$
$$0 \leqslant v_{A11} \perp 0.75 - (s_{A1} + s_{B1} - \tau_{A11}) \geqslant 0,$$
$$0 \leqslant v_{A21} \perp 0.75 - (s_{A2} + s_{B2} - \tau_{A21}) \geqslant 0,$$
$$0 \leqslant v_{B11} \perp 0.75 - (s_{A1} + s_{B1} - \tau_{B11}) \geqslant 0,$$
$$0 \leqslant v_{B21} \perp 0.75 - (s_{A2} + s_{B2} - \tau_{B21}) \geqslant 0,$$
$$0 \leqslant \tau_{A11} \perp s_{A1} - v_{A11} + \tau_{A11} \geqslant 0,$$
$$0 \leqslant \tau_{A21} \perp s_{A2} - v_{A21} + \tau_{A21} \geqslant 0,$$
$$0 \leqslant \tau_{B11} \perp s_{B1} - v_{B11} + \tau_{B11} \geqslant 0,$$
$$0 \leqslant \tau_{B21} \perp s_{B2} - v_{B21} + \tau_{B21} \geqslant 0.$$

Table 1 shows five different solutions representing alternative equilibria under these assumptions. The variable names have all been defined above, except for a_{A1}, a_{A2}, a_{B1}, and a_{B2}, which are the firms' marginal sales revenues at the two nodes calculated according to (5). All five solutions yield the same prices to consumers, but they have various amounts of sales, generation, and profits for the producers, as well as varying transmission prices. Equilibria I and II represent symmetric primal equilibria, in that the sales and generation by the two firms are equal. Comparing the two equilibria reveals the range of alternative transmission prices that can arise because firms A and B both explicitly recognize the joint transmission constraint in their profit maximization problems. In equilibrium I, the transmission fee w^* charged by the ISO is the highest possible among these symmetric solutions: 0.15. This means that the shadow prices for each firm's internal transmission constraint are the smallest possible: $\varphi_{A2} = 0$ and $\varphi_{B2} = 0.1$. Because φ_{A2} cannot be negative, a higher w^* (and thus a higher λ, which equals w^* in this case) would force the right side of s_{A2}'s complementarity condition in (9) to be strictly positive; this would be inconsistent with s_{A2} remaining positive. These solutions indicate that when the transmission fee is this high, generator A is indifferent at the margin about selling to node 2—if it could expand its sales there, its profit would not increase (indicated by $\varphi_{A2} = 0$). On the other hand, this is not true for company B. Because

Table 1. Five alternative equilibria for the example problem.

Variables	Equilibria I	II	III	IV	V
s_{A1}	0.375	0.375	*0.15*	*0.5*	*0.15*
s_{A2}	0.25	0.25	*0*	*0.5*	*0*
s_{B1}	0.375	0.375	*0.6*	*0.25*	*0.6*
s_{B2}	0.25	0.25	*0.5*	*0*	*0.5*
g_A	0.625	0.625	*0.15*	*1*	*0.15*
g_B	0.625	0.625	*1.1*	*0.25*	*1.1*
ψ_A	0	0	0	0	0
ψ_B	0	0	0	0	0
λ	0.15	*0*	0.15	0.15	*0.25*
w	0.15	*0*	0.15	0.15	*0.25*
φ_{A2}	0	*0.15*	0	0	0
φ_{B2}	0.1	*0.25*	0.1	0.1	*0*
a_{A1}	0.1	0.1	0.1	0.1	0.1
a_{A2}	0.25	0.25	0.25	0.25	0.25
a_{B1}	0	0	0	0	0
a_{B2}	0.25	0.25	0.25	0.25	0.25
v_{A11}	0.15	0.15	0.15	0.15	0.15
v_{A21}	0	0	0	0	0
v_{B11}	0.25	0.25	0.25	0.25	0.25
v_{B21}	0	0	0	0	0
τ_{A11}	0	0	0	0	0
τ_{A21}	0	0	0	0	0
τ_{B11}	0	0	0	0	0
τ_{B21}	0	0	0	0	0
p_1	0.25	0.25	0.25	0.25	0.25
p_2	0.25	0.25	0.25	0.25	0.25

Notes. Italics indicate values that differ from equilibrium I.
I: Firms' sales equal, highest possible transmission fee w^*.
II: Firms' sales equal, lowest possible transmission fee w^*.
III: Highest possible sales for B.
IV: Highest possible sales for A.
V: A solution from the restricted-multiplier formulation.

its marginal generation cost is less than A's, it would earn more profit if it could expand sales to node 2. In particular, its profit would increase by $\varphi_{B2} = 0.1$, on the margin, per unit of additional transmission capacity that is made available.

On the other hand, equilibrium II represents the situation in which the ISO's transmission fee w^* is zero, so that the ISO earns no revenue, while the internal transmission shadow prices are high ($\varphi_{A2} = 0.15$ and $\varphi_{B2} = 0.25$). The generators pay nothing to the ISO for the transmission services they receive, while their sales revenue and generation costs are unchanged. Thus, this equilibrium is more profitable for each of them than equilibrium I; so, if they have a choice, they would choose this equilibrium. In practice, they can accomplish this by imposing an internal joint constraint that is slightly tighter than the ISO's constraint (i.e., $\alpha = T - \varepsilon$). As a result, the ISO's shadow price λ will be zero and so will be the transmission fee w^*. Oren (1997) has previously pointed out that this strategy and the resulting solution are optimal for the more sophisticated MPEC/EPEC game described in §1. Although this is true for the two-node case we consider here and for Oren's three node case, Stoft (1999) shows that this is not necessarily optimal for a general network. For instance, some generators may earn higher profits if w is nonzero if their generators and sales are located such that they are relieving rather than exacerbating binding transmission constraints (i.e., providing so-called "counterflows"). In that case, the w they pay is negative, and they would prefer nonzero values.

Turning to equilibria III and IV, they represent asymmetric primal outcomes. Equilibrium III represents the highest possible sales and profits at each node for company B. At node 1, instead of splitting the maximum possible sales at the price cap equally ($0.75/2 = 0.375$ each), B sells 0.6 and A only sells 0.15. This represents the maximum possible equilibrium sales by B because if it sold more ($s_{B1} > 0.6$), A would not respond by selling $s_{A1} = 0.75 - s_{B1}$. Instead, A would sell more, which would cause the price to drop below the price cap; B's optimal response would then be to shrink its sales so that total sales fall back to 0.75, allowing the price to rise again to the price cap. For node 2, it is not the price cap that results in alternative equilibrium sales, but the joint transmission constraint. In equilibrium III, B is in the lucky position of having all the transmission capacity to itself, so it is responsible for the entire sales $S_2 = 0.5$ at node 2.

The other extreme asymmetric primal equilibrium is equilibrium IV, where A has its maximum possible sales at each node. The most it can sell at node 1 in equilibrium is 0.5 out of the total price-capped sales S_1 to that node of 0.75. If it sold any more, B would prefer to expand sales so that the price falls below the cap, similar to the reverse situation just explained for equilibrium III. The upper bound to A's sales there (0.5) is less than B's upper bound (0.6, equilibrium III) because B has a lower marginal cost, which translates into a greater willingness to expand sales and cut prices. Considering node 2, A's maximum equilibrium share of the limited transmission capacity is 0.5, the entire capacity. Thus, equilibrium III and IV show that any split of the transmission capacity between the two generators is an equilibrium. This, however, is not a general result; it depends on the slopes of the demand curves, size of the transmission limit, and marginal generation costs.

Equilibrium V, the last solution in Table 1, is a solution from the restricted-multiplier formulation of §4.1. This forces each company to have the same multiplier φ_2 for the sales cap constraint for node 2 (equivalent here to the transmission constraint between 1 and 2), rather than separate φ_{A2} and φ_{B2}. The only value of φ_2 for which that is possible is $\varphi_2 = 0$, which occurs if the transmission fee $w^* = 0.25$. Only firm B finds it profitable to sell at node 2 in the restricted-multiplier formulation, and market shares of the two firms are the same as in equilibrium III. Thus, restricting the multipliers has eliminated any solutions in which the more costly firm A sells power at the node with the sales constraint. However, there are still multiple equilibria at node 1; equilibrium V is just one of the possibilities. Other possibilities give A a greater share of the market at node 1, as much as $s_{A1} = 0.5$ (just as in equilibrium IV). Therefore, restricting the multipliers as in §4.1 can eliminate multiple solutions that result from the joint sales constraint, but not multiple solutions that arise because of price caps or other piecewise linearities that are modeled in the manner described in §2.2.

6. Concluding Remarks

A generalization of a standard Cournot model of competition among electricity generators on a transmission network has been presented that includes in each generator's profit maximization problem two new features: (1) a set of joint constraints involving other producers' decision variables, and (2) piecewise-linear demand curves. These extensions have important economic applications because, for example, liberalized electricity markets often are subject to price caps that transform affine demand curves into piecewise linear ones. This generalization presents analytical and computational challenges, some of which have been addressed in this paper.

An open analytical question concerns whether it is possible to show that consumer prices and consumption levels might be unique in these models, and under what circumstances. This is true for the simple example in §5, even though the piecewise-linear demand curves and joint transmission constraints mean that producer outputs and profits and transmission operator revenues are not unique. A remaining computational question is whether alternatives to Lemke's algorithm might perform better in uncovering alternative equilibria. It has been shown in this paper that successful application of that algorithm is only assured if a restricted formulation of the generalized model is applied in which each firm's set of multipliers for the joint constraints are constrained to be equal. This can be a reasonable assumption if the joint constraint represents a resource that the producers compete for in a market process, and if each producer is a price-taker with respect to the resource prices. However, this assumption is difficult to defend in general, so alternative algorithms which do not require that assumption are desirable.

The restricted LCP formulation that we have employed to treat the joint sales constraints $S_i \leqslant \sigma_i$, $i \in \mathcal{N}$, can be extended to deal with other similar constraints that couple the firms' variables, provided that these constraints are common to all the firms' problems. For problems where the firms have distinct joint constraints, it is no longer clear whether the restricted LCP approach remains applicable to compute an equilibrium solution. Such problems are instances of a generalized Nash game in its broadest form; for more discussion on the latter game, we refer the reader to Pang (2003) and Pang and Fukushima (2005).

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